RIGID MAXIMALLY MUTABLE LAURENT POLYNOMIALS

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Fano varieties

X Fano $\rightsquigarrow$ $-K_X$ ample

positively curved

examples — $\mathbb{P}^N$, $w\mathbb{P}^N$

basic building blocks — MMP

└— explicit constructions

There are finitely many Fano varieties, in each dimension, up to deformation

Kollár–Miyaoka–Mori

Birkar

Smooth Fano varieties:

$d=1$	1
$d=2$	10
$d=3$	105
$d>3$	???

Singular Fano varieties

???

Mirror Symmetry

Golyshev, Coates - Corti - Galkin - Golyshev - Kasprzyk

	MIRROR SYMMETRY	
Fano manifolds	$\longleftrightarrow$	Laurent polynomials

Question: What class of Laurent polynomials corresponds to Fano varieties?

Today:

- Give a conjectural answer to this question, which works in all dimensions
- Give evidence for the conjecture

Mirror symmetry, in more detail

X Fano $\rightsquigarrow$ $G_X(t) = 1 + \sum_{d=2}^{\infty} c_d t^d$

quantum period

$c_d = \langle [\mathrm{vol}]\, \psi^{d-2} \rangle_{0,1,d}$

Gromov-Witten invariant

The quantum period is a solution to the quantum differential equations for X.

$f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ $\rightsquigarrow$ classical period

$$\pi_f(t) = \frac{1}{(2\pi i)^n} \int_{(S^1)^n} \frac{1}{1 - tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$= \sum_{n=0}^{\infty} c_n' t^n \qquad c_n' = \mathrm{coeff}_1 (f^n)$$

The classical period is a solution to the Picard-Fuchs equations

Mirror symmetry, for us: consider the regularised quantum period

$$\hat{G}_X(t) = 1 + \sum_{d=2}^{\infty} c_d \, d! \, t^d$$

Then f is a mirror partner to X iff

$$\boxed{\hat{G}_X = \pi_f}$$

Fano manifolds Laurent polynomials

Mirror theorems by Givental and others provide Laurent polynomial mirrors to many Fano varieties.

Mutation

$$\pi_f = \frac{1}{(2\pi i)^n} \int_{(S^1)^n} \frac{1}{1-tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

The classical period is preserved by special changes of variables:

$$x_i \longmapsto \prod_j x_j^{m_{ij}} \qquad (m_{ij}) \in GL(n, \mathbb{Z})$$

$$x_i \longmapsto \begin{cases} x_i & 1 \leq i \leq n-1 \\ A(x_1, \ldots, x_{n-1})\, x_n & i = n \end{cases}$$

$\uparrow$ LAURENT POLYNOMIAL

$w = (000 \cdots 01)$

$h = A$

Galkin – Usnich

Akhtar – Coates – Galkin – Kasprzyk

Mutation

More invariantly : $f \in \mathbb{C}[N]$, $M = N^\vee$

$w \in M$ primitive

$h \in \mathbb{C}[w^\perp]$

$$\mu : \mathbb{C}(N) \longrightarrow \mathbb{C}(N)$$
$$x^\gamma \longmapsto h^{w(\gamma)} x^\gamma$$

CLUSTER TRANSFORMATION

Fomin - Zelevinsky, Fock - Goncharov

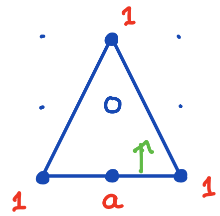
Lam - Pylyavskyy

Gross - Siebert, Gross - Hacking - Keel

Note that if f is a Laurent polynomial and μ is a mutation then $\mu(f)$ will in general not be a Laurent polynomial

Example

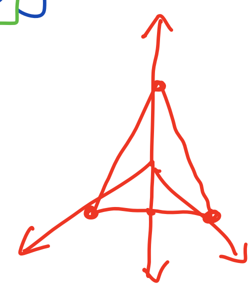
$$f = y + \frac{1}{xy} + \frac{a}{y} + \frac{x}{y}$$



$w = (0\ 1)$

$$\mu : \begin{cases} x \longmapsto x \\ y \longmapsto \underbrace{\boxed{(1+x)}}_{h}\, y \end{cases}$$

$$f = y + \frac{1+ax+x^2}{xy}$$



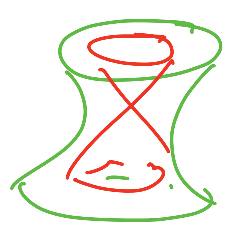
$$\mu(f) = (1+x)y + \frac{1+ax+x^2}{x(1+x)y}$$

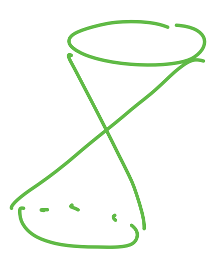
q^{β}

$\rightsquigarrow q^{-k_x \cdot \beta}$

In general this is not a Laurent polynomial. if $a=2$

$x^2+y^2=z^2+\lambda w^2$ $\quad (1+x)y + \dfrac{(1+x)^2}{x\cancel{(1+x)}y}$





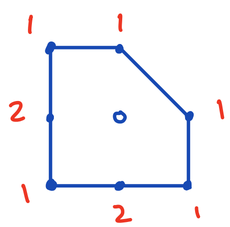
$\lambda \to 0$

Characterizing LP mirrors to Fano manifolds

First attempt: binomial coefficients on edges

Galkin, Przyjalkowski

There are 16 reflexive polygons, up to $GL(2,\mathbb{Z})$.



Putting binomial coefficients on edges gives 16 LPs

$\rightsquigarrow$ 8 mutation families

Each is mirror to a smooth del Pezzo surface.

Missing: dP_1, dP_2

$-K_X$ ample but not very ample

$X \longleftrightarrow f$

Fano

Laurent polynomial

$\rightsquigarrow$ $P = \mathrm{Newt}(f)$

$\downarrow$

Spanning fan of P

$\rightsquigarrow$ toric variety X_f

expect $X \rightsquigarrow X_f$

This doesn't work very well in 3 dimensions

4319 reflexive polytopes

→ mirrors to 92 of the 105 smooth Fano 3-folds

→ more than 2000 LPs that are not mirror to any smooth Fano 3-fold.

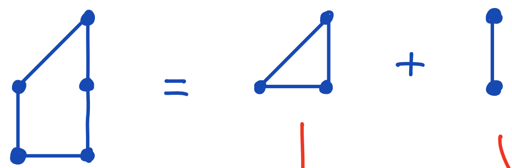
Second attempt: Minkowski polynomials

Coates – Corti – Galkin – Golyshev – Kasprzyk

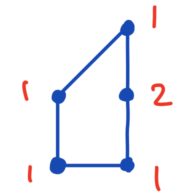
Akhtar – Coates – Galkin – Kasprzyk

Consider 3D reflexive polytopes

Look at Minkowski factorizations of facets (cf. Altmann) example



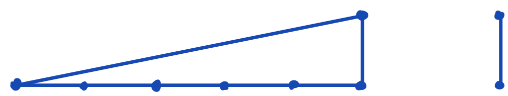
$$(1+x+xy)(1+y)$$



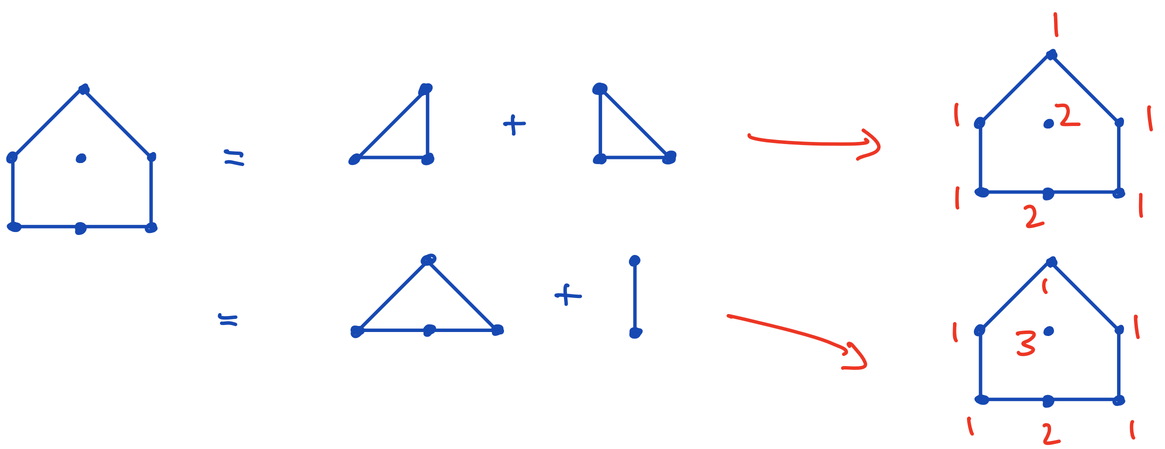
$P+Q$

$= \{p+q : p\in P, q\in Q\}$

Look at Minkowski factorizations into A_n-triangles



example



Factorizations may not exist, and may not be unique.

Minkowski polynomials

4319 reflexive polytopes

$\rightsquigarrow$ ~3800 Laurent polynomials

$\rightsquigarrow$ 165 mutation families

$$165 = \underbrace{98}_{} + 67$$

smooth Fano 3-folds with $-K_X$ very ample

Drawbacks:

- only applies to reflexive polytopes
- only applies in dimension 3
- 67 "extra" classical periods
- not closed under mutation

Maximally mutable Laurent polynomials

Start with any Fano polytope

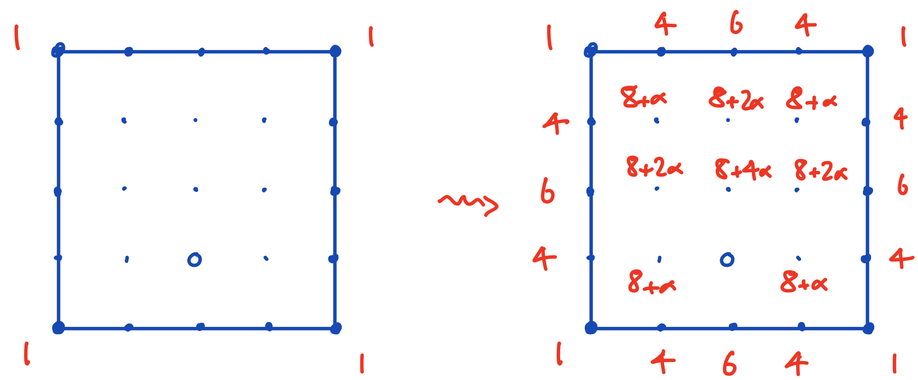
contains origin in strict interior
primitive vertices

Put 1s on the vertices $\mathrm{Newt}(f) \rightsquigarrow f$

Insist that f is compatible with as many mutations as possible

can read possible mutations off of $\mathrm{Newt}(f)$

Example



$w = (0\ 1)$

$h = (1+x)^4$

Maximally mutable Laurent polynomials with no free parameters are called <u>rigid</u>.

$$x^\gamma \longmapsto x^\gamma h^{w(\gamma)}$$

$h \in \mathbb{C}[w^\perp]$

require h has 1s on vertices and coefficients in $\mathbb{N}$

Mutation graph

Want to regard monomial changes of variables as trivial, i.e. study Laurent polynomials $f \in \mathbb{C}[N]$ up to $GL(N)$. But we need to be careful with automorphisms.

Mutation: $\mu_{w,h} : \mathbb{C}(N) \to \mathbb{C}(N)$ $\quad w \in M$

$x^\gamma \mapsto h^{w(\gamma)} x^\gamma$ $\quad h \in \mathbb{C}[w^\perp]$

Will identify $\mu_{w,h}$ with $\mu_{w, \underbrace{x^a}h}$ if $a \in w^\perp$

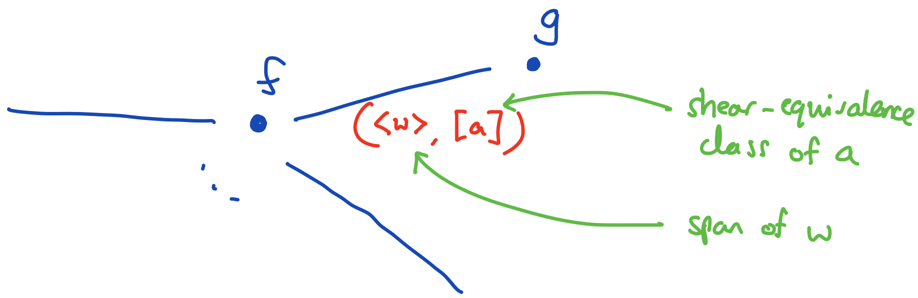
identify mutations that differ by a shear

Put 1s on vertices of all Newton polytopes

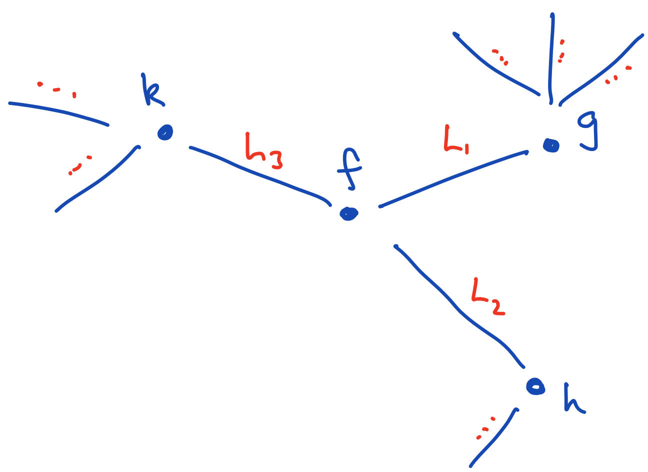
Take all Laurent polynomials to have coefficients in $\mathbb{N}$.

Start with f

Look at mutations $\mu_{w,h}(f) = g$ of f



Repeat!



$\mathcal{G}_f$ is defined by:

- removing the edge labels
- replacing the vertex label g by the $GL(N)$-equivalence class of $\mathrm{Newt}(g)$.

Specialising coefficients from F to f gives more mutations, i.e.

$$\mathcal{G}_F \hookrightarrow \mathcal{G}_f$$

We say that F is maximally mutable iff $\mathcal{G}_F$ is maximal.

Conjecture

Coates - Kasprzyk - Pitton - Tveiten

Corti, Golyshev

Rigid MMLPs up to mutation	$\longleftrightarrow$	Fano varieties with terminal locally toric qG-rigid singularities up to deformation
n = # variables		n = dimension of Fano variety

Note that in dimension ≤ 3, terminal Gorenstein qG-rigid $\Rightarrow$ smooth

Results

dimension 2

Kasprzyk-Nill-Prince classified all polygons that could admit a rigid MMLP.

We give a precise characterisation of MMLPs in two variables

$\rightsquigarrow$ exactly 10 mutation families of rigid MMLP in two variables

$\updownarrow$ MIRROR SYMMETRY

smooth del Pezzo surfaces

dimension 3

computer-assisted classification of rigid MMLPs with Newt(f) reflexive

$\longrightarrow$ 98 mutation families

$\updownarrow$ MIRROR SYMMETRY

smooth Fano 3-folds with $-K_X$ very ample

Systematic search beyond reflexive case $\longrightarrow$ exactly 7 more families

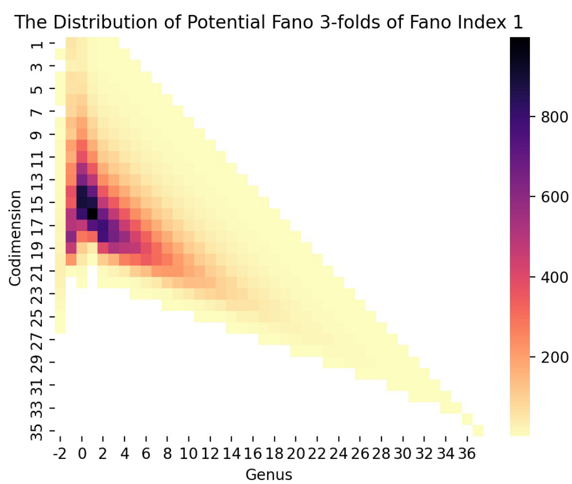
$$98 + 7 = 105$$

Beyond Gorenstein

Coates - Heuberger - Kasprzyk - Pitton

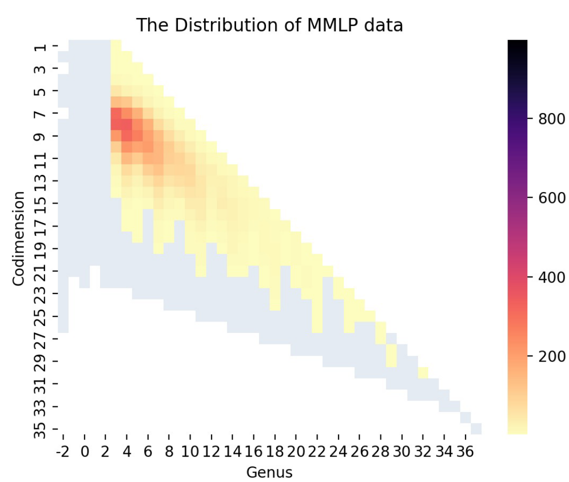
Find rigid MMLPs with Newt(f) a 3-dimensional canonical polytope.

8301 mutation families (Fano index 1)



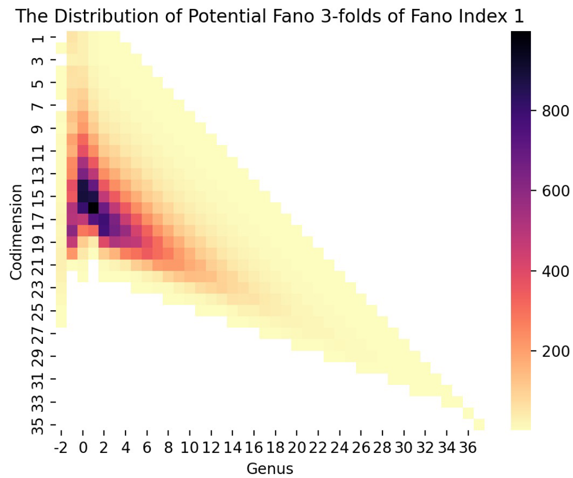
GRDB data

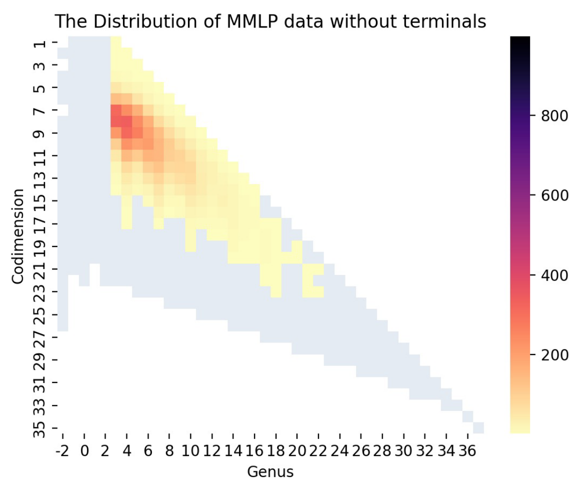
(Hilbert series only)

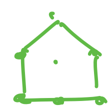


MMLP data

($\mathbb{Q}$-Fano 3-folds, conjecturally)

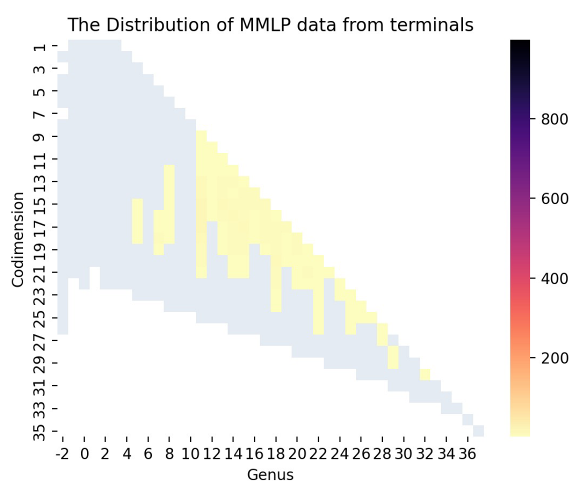

The Distribution of Potential Fano 3-folds of Fano Index 1
Codimension
Genus
800
600
400
200

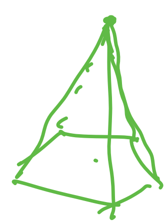

The Distribution of MMLP data without terminals
Codimension
Genus
800
600
400
200



$z=-1$

$(0,0,1)$


The Distribution of MMLP data from terminals
Codimension
Genus
800
600
400
200



$z=-1$