Reconstructing curves from their Hodge classes (j.w.w. Pirola - Schlesinger) $/\mathbb{C}$

mercoledì 12 aprile 2023 10:45

$X \subseteq \mathbb{P}^{2n+1}$ smooth hypersurface $\dim X = 2n$, $d = \deg X$

GRIFFITHS $\left\{ \begin{array}{c} \text{prim. cohom. classes} \\ H^{p,q}(X)_{prim} \end{array} \right\} \xleftrightarrow{\simeq} \left\{ \begin{array}{c} \text{certain degree} \\ R(X) \end{array} \right\}$

$$R(X) = \bigoplus_n \frac{H^0 \mathcal{O}_{\mathbb{P}^{2n+1}}(n)}{J_{X,n}}$$

J → Jac. ideal gen by partial derivatives of f, $X = \{f = 0\}$

e.g. $p = q = n$; $Y \subseteq X$ (smooth) subvariety of dim n

$[Y] \in H^{n,n}(X)$ $\quad H^{n,n}(X)_{prim} \simeq R_{(n+1)d-2n-2}$

$[Y] \longleftrightarrow I_{\alpha_Y} \subseteq R(X)$ ideal (Voisin + ... ; more algebraic version E-P)

PROPERTIES: • (J_X), $I_Y \subseteq I_{\alpha_Y}$

RECONSTRUCTION: Under which conditions $I_Y = I_{\alpha_Y}$?

($d = \deg X >> \deg Y$)

• There could be $Y_1 \subseteq X$ (smooth) subv. of dim n with $I_{\alpha_Y} = I_{\alpha_{Y_1}}$

$I_{Y_1} \subseteq I_{\alpha_Y}$

GENERALIZATION OF RECONSTR.: (Movasati - Sertöz PERFECTION)

Def $[Y]$ is PERFECT (at degree m) if $\exists\, Y_1, \dots, Y_k \subseteq X$

with $I_{\alpha_Y} = I_{\alpha_{Y_i}}$ $\forall i = 1, \dots, k$

$$I_{\alpha_Y, m} = I_{Y,m} + \sum_{i=1}^{k} I_{Y_i, m}$$

Q: (MS) Are all the classes perfect? (reconstructible)

Examples: • Y is a complete intersection $\Rightarrow$ perfect.

COUNTEREXAMPLES: • (CPS) C smooth quartic rational curve $\subseteq S$ smooth quartic surface $\subseteq \mathbb{P}^3$

$$I_{C,3} \,(+ I_{D,3} + J_{X,3}) \subsetneq I_{\alpha_{C,3}}$$

$\cdot (n \geq 2)$ Franco-VL: $Y \subseteq X$ of degree $d = 3, 4, 6$

$n=1$ CURVES: $C \subseteq S \subseteq \mathbb{P}^3$ (Ellingsrud-Peskine)

NL: $d \geq 4$, general S has Picard rank $= 1$

NL_d = surfaces of degree d with P. rk ≥ 2

$H^{1,1}(X)_{prim} \cong R_{2d-4}$

$$0 \to \mathcal{O}_S(-d) \to \Omega^1_{\mathbb{P}^3|S} \to \Omega^1_S \to 0$$ exact sequence

taking cohomology

$$H^1\Omega^1_{\mathbb{P}^3|S} \longrightarrow H^1\Omega^1_S \longrightarrow H^2\mathcal{O}_S(-d) \cong H^0\mathcal{O}_S(2d-4)^*$$

$H^1\Omega^1_{\mathbb{P}^3|S} \cong \mathbb{C}\langle \eta_H \rangle$, $H^{1,1}(X) \to H^1\Omega^1_S$

$(c) \mapsto \alpha_c : H^0\mathcal{O}_S(2d-4) \to \mathbb{C}$

$H^1\Omega^1_S / \mathbb{C}\langle \eta_H \rangle \cong H^{1,1}(S)_{prim} \to 0$

$H^0_*\mathcal{O}_S \supseteq I_{\alpha_c} = \mathrm{Ann}(\alpha_c)$ $\quad [\, I_{\alpha_c, m} = \{ f \in H^0\mathcal{O}_S(m) \mid \alpha_c(f \cdot g) = 0 \ \forall g \in H^0\mathcal{O}_S(2d-4-m) \}]$

$\alpha_c = 0$ iff $C \sim tH$

CONSTRUCTIVE METHOD:

$$0 \to N_{C|S} \cong \omega_C(4-d) \to N_C \to N_{S|C} \cong \mathcal{O}_C(d) \to 0$$

twist by $-d$ + take cohomology

$1 \mapsto \beta_C : H^0\mathcal{O}_C(2d-4) \to \mathbb{C}$

$$H^0N_C(-d) \xrightarrow{b} H^0\mathcal{O}_C \xrightarrow{\delta} H^1\omega_C(4-d) \cong H^0\mathcal{O}_C(2d-4)^*$$ exact seq.

$\boxed{\mathrm{Ann}(\beta_C)}$

PROP:
- $\mathrm{Ann}\,\beta_C \longleftrightarrow \mathrm{Im}\, b$
- $\boxed{\alpha_C = \pi^*\beta_C}$
- $\pi^{-1}(\mathrm{Ann}\,\beta) \subseteq I_{\alpha_y} = \mathrm{Ann}(\alpha_C)$
  $=$ iff π surjective

$$H^0\mathcal{O}_S(m) \xrightarrow{\pi} H^0\mathcal{O}_C(m)$$

$$H^0N_C(m-d) \xrightarrow{b} H^0\mathcal{O}_C(m) \xrightarrow{\delta} H^1\omega_C(4-d-m)$$

Rmk. π surj $\forall m$ iff$^{(2d-4)}$ C ACM curve ($H^1\mathcal{I}_C(n) = 0 \ \forall n$)

- For instance, C c.int. $\Rightarrow C$ ACM
  C twisted cubic $\Rightarrow C$ ACM

• $\pi^{-1}(\mathrm{Ann}\,\beta) \overset{\downarrow}{\subseteq} I_{\alpha_C}$

• $\mathrm{Ker}\,\pi = I_C = H^0_*\mathcal{J}_C$

$\Rightarrow I_C \subseteq I_{\alpha_C}$

$(=)$: $\boxed{\begin{array}{l} * : h^0 N_C(m-d) = 0 \\ * \ \ h^1 \mathcal{J}_C(2d-4-m) = 0 \end{array}}$ $\Rightarrow I_{C,m} = I_{\alpha_C, m}$

m := min degree in which I_C is generated (for ex before C quartic & $m=3$)

MORE (F?): C ACM curve $\Rightarrow$ perfect (CPS)

Q: $\forall n$ ACM subvarieties are perfect?

COUNTEREX : C, we know $I_{C,3} \ni S$ quartic surface

we study condition $I_{\alpha_C} = I_{\alpha_D}$: $\exists\, s,t,p \in \mathbb{Z}$ rel. prime s.t. $\neq 0$
$sH + tC + pD \sim 0$

$\rightsquigarrow$ find only one other curve

D : smooth quartic curve

$$(J_{S,3}) + I_{C,3} + I_{D,3} \subsetneq I_{\alpha_C,3}$$

$\boxed{n>1}$

• $[Y]$ should be algebraic

• we can define $I_{\alpha_Y}, \mathrm{Ann}(\beta_Y)$

↳ more vanishing conditions

ex more diff to find

• ACM subv ? (complete int ✓)

• $(n=1)$ conditions for (general) reconstruction?

• (F-VL): examples in dim $Y > 1$

$\boxed{HL_{d,2n}}$ = locus of hypsf of dim $2n$, and degree d

$$\begin{tikzcd} & H^0_*\mathcal{O}_X \arrow[d] & \\ H^0 N_Y(\ldots) \arrow[r] & H^0_*\mathcal{O}_Y \arrow[r, dashed, "\text{comp. of maps}"] & H^0\mathcal{O}_Y(\ldots)^\vee \end{tikzcd}$$

Green-Voisin : $d \geq 5$ the comp. of min. codim are the ones containing lines

NL_d

Otwinowska : $d \gg n$ $(2n)$ true

(F-VL) : for "special" components, GV is true except for $d = 3,4,6$

(F-VL) : for "special" components, GV is true except for $d = 3,4,6$

[Y] "<u>fake</u> linear cycles" $\rightsquigarrow$ I_{α_y} is not <u>perfect</u>