Newton-Okounkov bodies & minimal models for cluster varieties

w/ Lara Bossinger, Man-Wai Cheung & Timothy Magee

To define a pair of cluster varieties of dimension r we need:

- A quiver Q with r vertices s.th.  and  are not subgraphs of Q.
- A subset $F \subseteq Q_0$ of "frozen vertices".

$$\begin{tikzcd} & \mathcal{A}_{Q,F} \\ Q,F \arrow[ur, bend left] \arrow[dr, bend right] & \\ & \mathcal{X}_{Q,F} \end{tikzcd}$$

Toric setting $Q = \bullet$

$\mathcal{A}_{Q,F} = T_N$ $\quad$ $\mathcal{X}_{Q,F} = T_M$

Rough description:

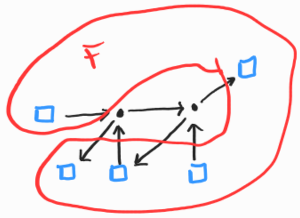
- $N \simeq \mathbb{Z}^r$ & $M := \mathrm{Hom}(N, \mathbb{Z})$
- $T_N := N \otimes \mathbb{C}^*$ & $T_M := M \otimes \mathbb{C}^*$

Then

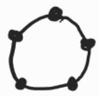
$$\mathcal{A}_{Q,F} = \bigcup_{s \in \Delta(Q,F)_0} T_{N,s}$$

$$\mathcal{X}_{Q,F} = \bigcup_{s \in \Delta(Q,F)_0} T_{M,s}$$

- $\Delta(Q,F)$ is a very specific simplicial complex associated to (Q,F), the cluster complex
- $\Delta(Q,F)_0$ are the vertices of $\Delta(Q,F)$
- each torus has preferred coordinates cluster coordinate
- change of coordinates is very specific. cluster transformatn

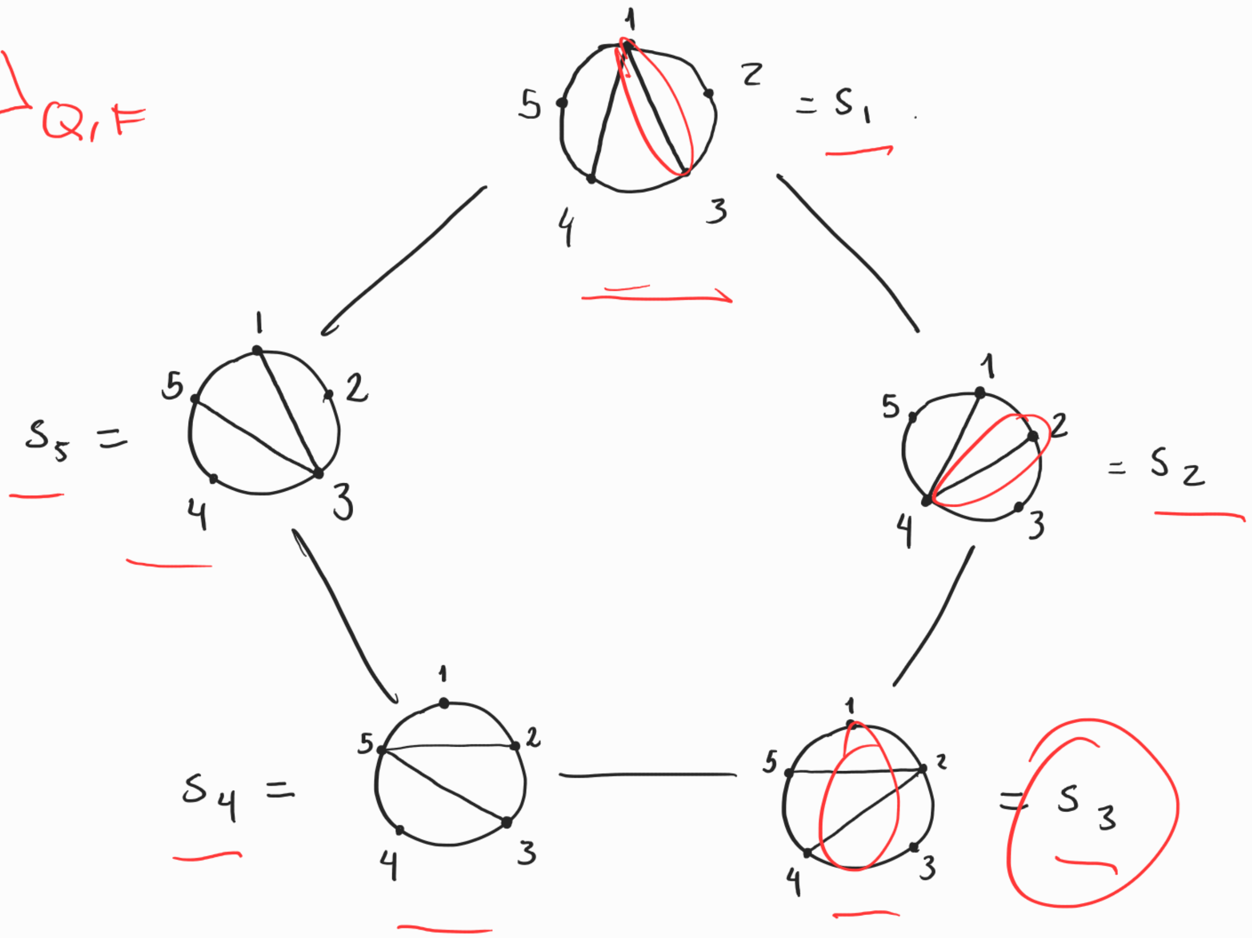
Example (Q,F): 

$r=7$

vertices of $\Delta(Q,F)$ are the triangulations of 

edges of $\Delta(Q,F)$ connect triangulations related by the "flip" of an arc.

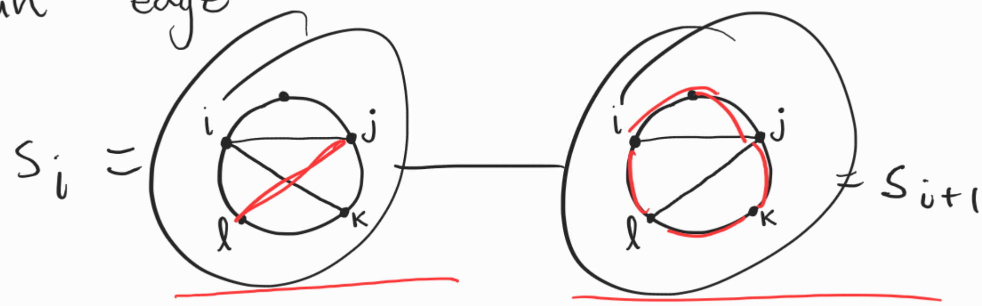
$\Delta_{Q,F}$



The torus T_{s_i} has coordinates

$$\{ P_{ij} \mid \overline{ij} \text{ is an arc of } s_i \}$$

For an edge



The change of coordinates is

$$\mathbb{C}(T_{s_{i+1}}) \dashrightarrow \mathbb{C}(T_{s_i})$$

$$P_{jl} \longmapsto \frac{P_{ij} P_{lk} + P_{il} P_{jk}}{P_{ik}}$$

In this case we have

- $\mathcal{A}_{Q,F} \overset{\text{codim } 2}{\simeq} \operatorname{Cone}(Gr_2(\mathbb{C}^5)) \setminus V\left(\prod_{i \in \mathbb{Z}_5} P_{i\, i+1} = 0\right)$
- $\mathcal{A}_{Q,F} \simeq \mathcal{X}_{Q,F}$

<u>Notation</u>

If $\mathcal{V} = \begin{cases} \mathcal{A}_{Q,F} \\ \mathcal{X}_{Q,F} \end{cases}$ then $\mathcal{V}^\vee = \begin{cases} \mathcal{X}_{Q,F} \\ \mathcal{A}_{Q,F} \end{cases}$

$\mathcal{V}$ and $\mathcal{V}^\vee$ are Fock-Goncharov dual or mirror dual. Write $\mathcal{V} = \bigcup_{s \in \Delta(Q,F)_0} T_{L,s}$ $\quad L = \begin{cases} N \\ M \end{cases}$

$\mathcal{V}$ has a canonical volume form $\Omega_{\mathcal{V}}$ such that:

$$\Omega_{\mathcal{V}}\Big|_{T_{L,s}} = \boxed{\frac{1}{z_1 \cdots z_r}}\, dz_1 \wedge \ldots \wedge dz_r \qquad \forall s \in \Delta(Q,F)_0$$

where $z_1, \ldots, z_r$ are the prefered coordinates of $T_{L,s}$.

<u>Conjecture</u> (Fock-Goncharov 03')

$\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ has a canonical basis parametrized by $\mathcal{V}^{\vee}(\mathbb{Z}^t)$ the integral tropical points of $\mathcal{V}^{\vee}$.

$$\mathcal{V}^{\vee}(\mathbb{Z}^t) = \left\{ \operatorname{ord}_D : \mathbb{C}(\mathcal{V}^{\vee})^* \longrightarrow \mathbb{Z} \;\middle|\; \begin{array}{l} D \text{ is a divisor on a variety} \\ \text{birational to } \mathcal{V}^{\vee} \text{ \& } \operatorname{ord}_D(\Omega_{\mathcal{V}}) < 0 \end{array} \right\}$$

- The Fock-Goncharov conjecture is false in general.
- In 2014 Gross-Hacking-Keel-Kontsevich introduced theta functions on cluster varieties and gave conditions ensuring that $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) = \bigoplus_{q \in \mathcal{V}^{\vee}(\mathbb{Z}^t)} \mathbb{C} \cdot \vartheta_q^{\mathcal{V}}$

<u>Example</u>

$$\left\{ \prod_{i \in \mathbb{Z}_5} P_{i\,i+1}^{c_i} \prod_{ij \in \operatorname{mut}(s)} P_{ij}^{a_{ij}} \;\middle|\; s \in \Delta(Q,F)_0 \,,\; a_{ij} \geq 0 \,,\; c_i \in \mathbb{Z} \right\}$$

is the set of theta functions on $\mathcal{A}_{Q,F}$.

Example If $\mathcal{V} = T_L$ correspons to $\boxed{Q = \bullet}$

then $\mathcal{V}^\vee = T_{L^*}$ and $\mathcal{V}^\vee(\mathbb{Z}^t) \equiv L^*$.

The canonical basis of $\Gamma(T_L, \mathcal{O}_{T_L})$ parametrized by L^* is the basis of characters.

Lemma A choice of torus $T_{L^*,s} \hookrightarrow \mathcal{V}^\vee$ gives rise to a bijection

$$\begin{aligned} \mathcal{V}^\vee(\mathbb{Z}^t) &\longrightarrow L^* \cong \mathbb{Z}^r \\ q &\longmapsto q_s \end{aligned}$$

We write $\mathcal{V}^\vee_s(\mathbb{Z}^t)$ to stress that we think of $\mathcal{V}^\vee(\mathbb{Z}^t)$ as the lattice L^* via such an identification. In particular, $\mathcal{V}^\vee_s(\mathbb{Z}^t) = L^* \hookrightarrow L^* \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$

Lemma $\mathcal{V}^\vee(\mathbb{R}^t)$ is well defined and every $s \in \Delta(Q,F)_0$ gives a bijection $\mathcal{V}^\vee(\mathbb{R}^t) \longrightarrow \mathbb{R}^r$. Moreover, different identifications are related by piece-wise linear isomorphisms.

$\boxed{\text{We always assume the FG conjecture holds}}$

In this case we can define the structure constants:

$$\vartheta_p \vartheta_q = \sum_{r \in \mathcal{V}^\vee(\mathbb{Z}^t)} \alpha(p,q,r)\, \vartheta_r$$

<u>Def</u> A closed subset $P \subseteq \mathcal{V}_s^\vee(\mathbb{R}^t)$ is <u>positive</u>, iff

$\forall\, a,b \in \mathbb{Z}_{\geq 0}$ $\forall\, p \in aP(\mathbb{Z}),\ q \in bP(\mathbb{Z})$

$\forall\, r$ s.th. $\alpha(p,q,r) \neq 0$ then $r \in (a+b)P$.

$$\boxed{\text{Every positive set } P \text{ determines a graded subring } R_P \subseteq \Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})[x].}$$

<u>Theorem</u> (GHKK 14' + Keel-Yu 19')

Let $P \subseteq \mathcal{V}_s^\vee(\mathbb{R}^t)$ be a top dimensional, compact, rational positive polytope. Then we have an inclusion:

$$\mathcal{V} \xhookrightarrow{\text{open}} \operatorname{proj}(R_P).$$

And a toric degeneration

Toric var assoc. to P

$$\big(\mathcal{V} \subseteq \operatorname{proj}(R_P)\big) \rightsquigarrow \big(T_L \subseteq TV_P\big)$$

Aim: ① Reverse this construction. Namely, for an open inclusion $\mathcal{V} \subseteq Y$ with Y projective construct a positive polytope $\dot{P}_Y \subseteq \mathcal{V}^\vee(\mathbb{R}^t)$.

Not always possible.

② When this is possible show that P_Y is a Newton-Okounkov body.

Let $\mathcal{V} \subseteq Y$ be a partial compactification.

Q: Can we obtain a basis for $\Gamma(Y, \mathcal{O}_Y)$ from the theta basis of $\Gamma(\mathcal{V}, \mathcal{O}_\mathcal{V})$?

Need Y to be sensible to the cluster structure.

Def-Lemma A partial minimal model of $\mathcal{V}$ is an open inclusion $\mathcal{V} \hookrightarrow Y$ into a normal variety Y such that $\Omega_\mathcal{V}$ has a simple pole along every irreducible component of $Y \setminus \mathcal{V}$. | minimal if Y is projective

Example Let frozen variables vanish.

We fix a p.m.m. $\mathcal{V} \hookrightarrow Y$ and let $D_1, \dots, D_n$ be the irreducible components of $Y \setminus \mathcal{V}$

$\mathrm{ord}_{D_i} \in \mathcal{V}(\mathbb{Z}^t)$

<u>Definition</u>

- The ϑ-<u>superpotential</u> asociated to $\mathcal{V} \subseteq Y$ is

$$W_Y = \sum_{i=1}^{n} \vartheta^{\mathcal{V}^\vee}_{\mathrm{ord}_{D_i}} \in \Gamma(\mathcal{V}^\vee, \mathcal{O}_{\mathcal{V}^\vee})$$

- We say $\mathcal{V} \subseteq Y$ has <u>enough</u> <u>theta</u> <u>functions</u> if $\{ \vartheta^{\mathcal{V}}_{\mathrm{ord}_{D'}} \mid \mathrm{ord}_{D'}(W) \geq 0 \}$ is a basis for $\Gamma(Y, \mathcal{O}_Y)$.

Intuitively, this is the set of ϑ-functions on $\mathcal{V}$ that extend to Y.

For it to be we need

$\forall\ a, b \in \mathcal{V}(\mathbb{Z}^t) \times \mathcal{V}^\vee(\mathbb{Z}^t)$ $\quad a(\vartheta^{\mathcal{V}}_b) = b(\vartheta^{\mathcal{V}}_a)$

&

$$b\Big(\sum_p c_p \vartheta^{\mathcal{V}^\vee}_p\Big) \geq 0 \iff b(\vartheta^{\mathcal{V}^\vee}_p) \geq 0$$

for all p s. th. $c_p \neq 0$.

Picture we are going for:

- Y is a normal proyective variety
- $\mathrm{Pic}(Y)$ is free of finite rank.
- $UT_Y = \mathbf{Spec}_Y\left(\bigoplus_{[\mathcal{L}] \in \mathrm{Pic}(Y)} \mathcal{L} \right)$

universal torsor

[If Y is smooth & $\mathrm{Cox}(Y)$ fin. gen $UT_Y(Y) = \mathrm{Spec}(\mathrm{Cox}(Y))$

If UT_Y is a partial minimal model of an $\mathcal{A}$-cluster variety with enough theta functions and the action of $\mathrm{Pic}(Y)^* \curvearrowright UT_Y$ is cluster then Y is a minimal model of $\mathcal{A}/T$ a cluster quotient of $\mathcal{A}$ and for every $[\mathcal{L}] \in \mathrm{Pic}\, Y$ we have a positive set $\Delta_{\mathcal{L}} \subseteq (\mathcal{A}/T_K)^\vee(\mathbb{Z}^t)$ such that $P_{\Delta_{\mathcal{L}}} \cong \bigoplus_{n \geq 0} \Gamma(Y, \mathcal{L}^n)$.

Moreover $\Delta_{\mathcal{L}}$ is a Newton-Okounkov body for a distinguished valuation on $\Gamma(\mathcal{A}/T_K, \mathcal{O}_{\mathcal{A}/T_K})$

Quotients and fibers of cluster varieties

Let $p : T_N \longrightarrow T_M$ be a monomial map.

the pull-back $p^* : \mathbb{C}[T_M] \longrightarrow \mathbb{C}[T_N]$ corresponds

to a homomorphism $p^* : N \longrightarrow M$

Let $K = \ker(p^*)$ then we obtain dual

maps $K \hookrightarrow N$ & $M \twoheadrightarrow K^*$

These correspond to

$$T_K \hookrightarrow T_N \qquad \& \qquad T_M \twoheadrightarrow T_{K^*}$$

If p^* corresponds to a matrix $B = (b_{ij})_{r \times r}$

such that $b_{ij} = (\# i \to j \text{ in } Q) - (\# j \to i \text{ in } Q)$

$\forall$ $i \in Q_0$ $j \in Q_0 \setminus F$ then p extends

to a map $p : \mathcal{A}_{Q,F} \longrightarrow \mathcal{X}_{Q,F}$

Moreover, we have maps

$$T_K \hookrightarrow \mathcal{A}_{Q,F} \qquad \& \qquad w : \mathcal{X}_{Q,F} \longrightarrow T_{K^*} \circlearrowright e$$

& $\mathcal{A}_{Q,F} / T_K$ is good quotient.

We obtain varieties that look like cluster varieties:

$$\mathcal{A}_{Q,F} / T_K = \bigcup_{s \in \Delta(Q,F)_0} \underline{T_{N/K,s}}$$

$$\underline{\mathcal{X}_e} = w^{-1}(e) = \bigcup_{s \in \Delta(Q,F)_0} \underline{T_{(N/K)^*,s}}$$

Let $\underline{(\mathcal{A}_{Q,F}/T_K)^\vee} = \underline{\mathcal{X}_e}$

<u>Example \ Theorem</u>

For $\mathcal{A}_{Q,F} \subseteq \mathrm{Cone}(Gr_2(\mathbb{C}^5))$ we can choose p^* such that the action $T_K \circlearrowright \mathcal{A}_{Q,F}$ coincides with the

$T_{P, C^*(Gr_2(\mathbb{C}^5))}$ - action on $\mathrm{cone}(Gr_2(\mathbb{C}^5))$

& $\mathcal{A}_{Q,F} / T_K \overset{\text{codim } 2}{\simeq}$ positroid variety inside $Gr_2(\mathbb{C}^5)$

Cluster valuations

We say that Q, F is of full rank if the matrix $B_{rec} = (b_{ij}) \in Mat(|Q_0| \times |Q_0 \setminus F|, \mathbb{Z})$ has full-rank

Theorem (in between the lines of GHKK - Fujita-Oya)

Suppose Q, F is of full-rank. Then for each $s \in \Delta(Q, F)_0$ there is a total order $\leq_s$ on M and a valuation

$$g_s : \Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \longrightarrow (\overset{\displaystyle M}{\underset{\displaystyle \mathcal{A}_s^\vee(\mathbb{Z}^t)}{\wr\wr}}, \leq_s)$$

such that $g_s(\vartheta_q) = q_s$ for every theta function

Corollary

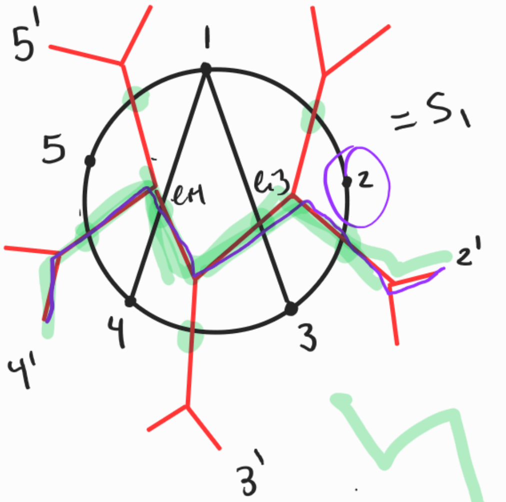
We have analogous valuations on $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ & on $\Gamma(\mathcal{A}/T_K, \mathcal{O}_{\mathcal{A}/T_K})$.

Remark

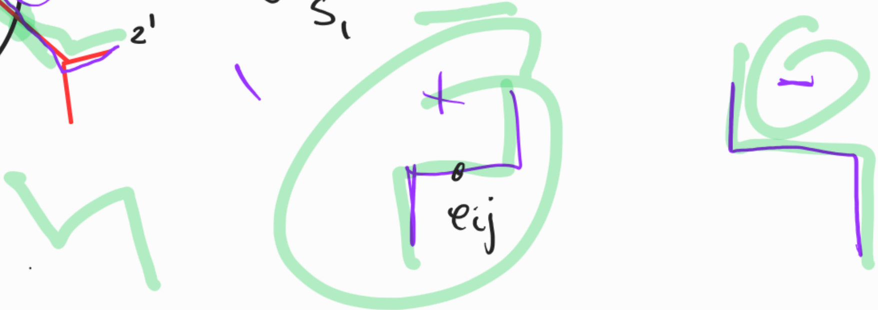
We are able to prove the existence of a valuation g_s beyond the full-rank case provided $\exists$ a GHKK degeneration.

Example $\mathcal{A}_{prm} \leftarrow \mathcal{A}$ $\quad M \simeq \mathbb{Z}^7 = \langle \underbrace{e_{12}, e_{13}, \dots, e_{15}}_{S}, e_{14}, e_{13} \rangle$

$\downarrow \mathbb{C}^n \qquad \qquad M \overset{P}{\leftrightarrow} N$



$$g_{s_1}(P_{24}) = e_{14} + e_{23} - e_{13}$$



Main theorem of the talk

Let Y be a normal projective variety such that $\mathrm{Pic}(Y)$ is free of finite rank and $\mathrm{Cox}(Y) := \Gamma(UT_Y, \mathcal{O}_{UT_Y})$ is fin. generated.

Definition

Let $\mathrm{val}: \mathrm{Cox}(Y) \longrightarrow (\mathbb{Z}^r, \leq)$ be a valuation.

The NO-body asoc. to $[\mathcal{L}] \in \mathrm{Pic}(Y)$ and val is

$$\Delta_{val}(\mathcal{L}) = \overline{\mathrm{conv}\left\{ \bigcup_{k \geq 1} \frac{\mathrm{val}(f)}{k} \,\middle|\, f \in \Gamma(Y, \mathcal{L}^{\otimes k}) \setminus \{0\} \right\}}$$

$$\subseteq \mathbb{Z}^r$$

Theorem (Bossinger – Cheung – Magee – NC)

Assume $\mathcal{A} \subseteq UT_Y$ is a partial minimal model with enough theta functions. Let

$$\{W^{trop}_{UT_Y} \geq 0\} := \{p \in \mathcal{A}^\vee(\mathbb{Z}^t) \mid p(W) \geq 0\}$$

Suppose that $\exists\ p : N \to M$ such that the action of $T_K \circlearrowright \mathcal{A}$ coincides with $T_{Pic(Y)^*} \circlearrowright UT_Y$.

In particular $T_{K^*} \cong T_{Pic}$

$$\mathcal{X} \xrightarrow{W} T_K \ni [\mathcal{L}] \otimes 1$$

Assume $\mathcal{A}$ has a g-vector valuation

Then for every $[\mathcal{L}] \in Pic(Y)$

$$\Delta_{g_s}(\mathcal{L}) = (w^{trop})^{-1}([\mathcal{L}]) \cap \{W^{trop}_{UT_Y} \geq 0\}$$

In particular $\Delta_{g_s}(\mathcal{L})$ is a positive set.

Moreover, if Δ_{g_s} & $\Delta_{g_{s'}}$ are connected to each other by iterated tropical $\mathcal{X}$-cluster transformations.

$\mathcal{L} \to \mathbb{P}^N$

- $\mathcal{A}/T_K \subseteq Y$ is a minimal model.

Remarks

- The theorem aplies e.g. to Grassmannians & Flag varieties
- We show that Rietsch-Williams' NO-bodies for Grassmannians are instances of this construction.
- Have other version of the construction for Weil divisors no reference to universal torsors.

THANKS!!!