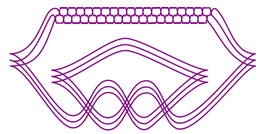
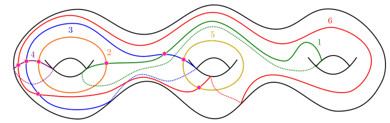
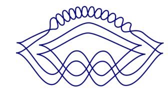
Positroid links & Braid varieties







(1) 2001.01334 w/ H. Gao

(2) 2007.04943 w/ E. Zaslow

R. Casals (UC Davis) @
Nottingham University Online AG Seminar

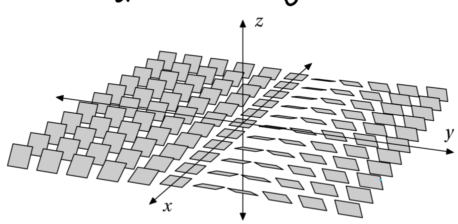
2009.06737 (3)

2101.02318 (4) w/ L. Ng

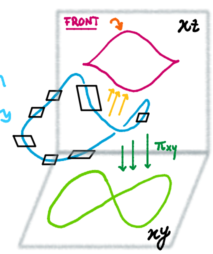
(5) 2012.06931 w/ E. Gorsky, M. Gorsky, J. Simental + upcoming!

§1. INTRODUCTION : LEGENDRIAN KNOTS in $\mathbb{R}^3$

Def^n: The standard contact structure on $\mathbb{R}^3$ is the 2-plane field $\xi_{st} := \operatorname{Ker}\{dz - y\,dx\}$. It compactifies to $(\mathbb{S}^3, \xi_{st})$. ∎



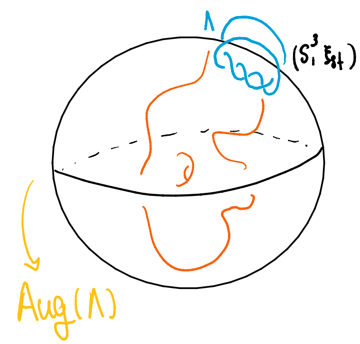
Legendrian knot



Def^n: A Knot $K \subseteq (\mathbb{R}^3, \xi_{st})$ is $\boxed{\text{LEGENDRIAN}}$ if $TK \subseteq \xi_{st}$. ∎

LAGRANGIAN FILLINGS : let $\Lambda \subseteq (S^3, \xi_{st})$ be a Leg. link.

Def^n: A Lagrangian filling $L \subseteq (\mathbb{D}^4, \omega_{st})$ is an embedded exact Lagrangian surface in $\mathbb{D}^4$ with boundary $\partial L = \Lambda$ in $\partial \mathbb{D}^4 = S^3$. ∎



Salient Facts:

(1) A Λ might or might not have a Lagr. filling.

(2) If $\exists L$ filling Λ then $\boxed{g(L) = g_s(\Lambda)}$.
different than smooth top.!

(3) (Eliashberg-Polterovich 1996) let $\Lambda = \Lambda_0$ be the max-tb standard unknot. Then $\exists!$ L filling (the flat disk) up to Hamiltonian isotopy.

(4) Lagr. fillings are the objects of $\mathcal{W}(\mathbb{C}^2, \Lambda)$, the wrapped Fukaya category stopped at Λ. (See also Sh_Λ.)

The Moduli of Lagrangian Fillings —— the mirror of the "LG Model" $(\mathbb{C}^2, \Lambda)$

Let $\Lambda \subseteq (S^3, \xi_{st})$ be Legendrian. A $\boxed{\text{Legendrian isotopy invariant}}$ (of some). bdy condition at ∞

Y. Chekanov '00, Ekholm-Etnyre-Sullivan, Ng, and more

$\boxed{\text{LEGENDRIAN CONTACT DGA } A_\Lambda}$

FLOER THY

(can be enhanced to a category Aug_+)

Let $L \subseteq (\mathbb{D}^4, \lambda_{st})$ be a Lagrangian filling, induces $\varepsilon_L : A_\Lambda \longrightarrow \mathbb{Z}[H_1(L)]$ "augmentation".

(*) $\boxed{\text{Augmentation variety } Aug}$. → cluster variety (if $\Lambda = \Lambda(\beta)$)

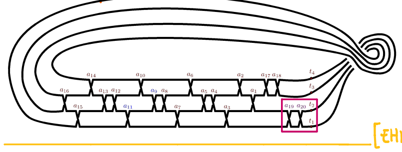
affine variety

holomorphic symplectic [C.: Gorsky^2-Sim.]

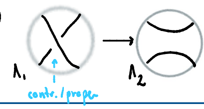
Two facts : $\boxed{\partial C_{ij} = \prod_{\text{\{cusps\}}} P_\gamma + Id}$ if 

(1) Differential Graded Algebra A_Λ

J-hol. strips. Maslov Reeb chords



[EHK, C.-Ng]

(2) 

induces explicit map $A_{\Lambda_1} \longrightarrow A_{\Lambda_2}$

§ 2. Braid varieties I ← a class of affine algebraic varieties which are useful to apply topology to AG!

Consider $B_i(z) := \begin{pmatrix} 1 & \cdots & & & 0 \\ & \ddots & & & \\ 0 & \cdots & \boxed{\begin{matrix} 0 & 1 \\ 1 & z \end{matrix}} & \cdots & 0 \\ & & & \ddots & \\ 0 & & \cdots & & 1 \end{pmatrix} \in GL_n(\mathbb{C}[z])$.

Def^n: Let $\beta \in \mathcal{B}_n$ be a positive braid word $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$. affine by def^n!

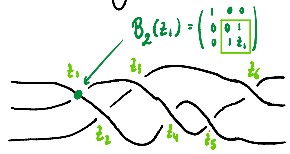
the braid variety associated to β is :

$$X(\beta) := \{(z_1, \dots, z_\ell) \in \mathbb{C}^\ell : B_{i_1}(z_1) \cdot \ldots \cdot B_{i_\ell}(z_\ell) \cdot w_0 \text{ is upper triangular}\} \subseteq \mathbb{C}^{\ell(\beta)}$$

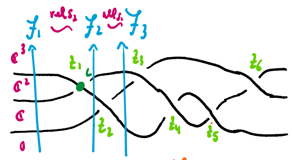
Example: $X(\sigma_1^4) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : 1 + z_1 z_2 + z_4(z_1 + z_3 + z_1 z_2 z_3) = 0\} = \mathbb{C}^3 \setminus \{z_1 + z_3 + z_1 z_2 z_3 = 0\}$,

↳ is this a friend? Yes! Foliated by $\{z_1 + z_3 + z_1 z_2 z_3 = \alpha\}$, $\alpha \in \mathbb{C}^\times$, which is A_2-cluster var $\mathbb{P}_5$.

What is $X(\beta)$ doing? ← the most correct answer is "Moduli Space of Augmentations for $\Lambda(\beta\Delta)$".

① counts contain PATHS 

$\beta = \sigma_2 \sigma_1 \sigma_2 \sigma_1^2 \sigma_2$

② 

} parametrizes OBS flags

$\beta = \sigma_2 \sigma_1 \sigma_2 \sigma_1^2 \sigma_2$

③ If β is an algebraic link, the coordinate ring of $X(\beta\Delta)$ is a cluster algebra (e.g. z_i are A-coord.)

↳ geometrizes the FPST cluster alg.

④ The cohomology $H^*(X(\beta))$ and its weight filtr. are interesting, ⇝ recover triply graded HOMFLY.

↳ see recent H^* computations by Galashin-Lam, (q,t)-Catalan

↳ brick mfld compactification (L. Escobar)

Thm: $X(\beta)$ is independent of the choice of braid word β in Br_n^+. ← * under RIII & Δ-conjugations

In fact $X(\beta\Delta)$ is smooth and $\exists$ free T-action s.t.

$X(\beta\Delta)/_T$ is smooth, holomorphic symplectic

and its coordinate ring admits a cluster A-structure.

$X \neq \phi$ iff $\delta(\beta) = w_0$, compl. inter. of dim $\ell(\beta) - \binom{n}{2}$

ABSOLUTE

$\boxed{\text{Example}^*: B_i(z_1) B_{i+1}(z_2) B_i(z_3) = B_{i+1}(z_3) B_i(z_2 - z_1 z_3) B_{i+1}(z_1)}$

RELATIVE

Thm: $\exists$ diagrammatic calculus to study the category of CORRESPONDENCES of BRAID VARIETIES.

The main protagonist is $\sigma_i \sigma_i \longrightarrow \sigma_i$ (nil Hecke move)

⇝ new stratifications, diagrams for cluster strata AND deep strata.

§3. Positroids in $Gr(k,n)$ — or, "dude, where is my braid?"

There exists a stratification $Gr(k,n) = \bigcup_{\substack{u,w \in S_n \\ u \leq w}} \mathring{\Pi}_{u,w}$, with a unique open stratum

proj. of Richardson $\mathring{R}^u_w := \mathring{X}_w \cap \mathring{X}^u \subseteq \mathcal{F}l_n$

$$\mathring{\Pi}_{k,n} := \{ V \in Gr(k,n) : \text{consecutive cyclic Plücker non-zero} \}.$$

Example: $(k,n) = (2,5)$, $\mathring{\Pi}_{2,5} = Gr(2,5) \setminus \{ \underbrace{\Delta_{12} \cdot \Delta_{23} \cdot \Delta_{34} \cdot \Delta_{45} \cdot \Delta_{51}}_{\text{in anticanonical class}} = 0 \}$.

with $w =$  and $u = id$.

The underlying braid is $\beta = wu^{-1} = \sigma_1\sigma_2\sigma_3\sigma_2\sigma_4\sigma_3$. $\rightsquigarrow$ $X(\beta\Delta) \cong \mathring{\Pi}_{2,5}$ $\rightsquigarrow$ a trefoil knot

Combinatorics to braids (to Leg. links, to DGAs, to ∞, and beyond)

$u, w \in S_n$, $u \leq w$, w is k-Grassmannian $\longleftrightarrow$ $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ affine k-bounded permutation $\longleftrightarrow$ Le Diagram for λ, λ partition in $k \times (n-k)$ box $\longleftrightarrow$ Cyclic rank matrix $r = (r_{ij})$, $i, j \in \mathbb{N}$

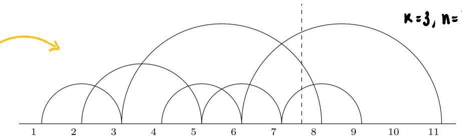
$\beta(w)\beta(u)^{-1} \cdot \Delta$
- not necessarily positive word
- n stranded

= Richardson braid = $R_n(u,w)$

juggling diagram
- positive word
- k stranded

= Juggling braid = $J_k(f)$

$k=3$, $n=7$



Example: $J_3(f)$ is good, $X(J_3(f)) \subseteq Gr(3,7)$.

But, $R_7(u,w) = (\underbrace{\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_6\sigma_5}_{w})(\underbrace{\sigma_2\sigma_3}_{u})^{-1}$.

negative crossings!

§4. Braid varieties II ← negative crossings (RII) & Markov stabilizations

$1 \leftrightarrow \sigma_i \cdot \sigma_i^{-1}$ $\qquad$ $\beta_1 \cdot \sigma_n \cdot \beta_2 \sim \beta_1\beta_2$ if $\beta_i \in Br_n$.

The two TAKE HOME NUGGETS are:

① Let $\eta \in \mathcal{B}_n$ be equiv. to a positive word β. Then $\exists$ affine variety $X(\eta)$ and a set of locally nil. derivations $V(\eta)$ s.t. $\boxed{X(\eta)/V(\eta) \cong X(\beta)}$.

← positive crossings give variables $\mathcal{B}_i(z_j)$

← negative crossings give $\mathbb{C}$-action (and $\mathcal{B}_i(0)$ to eq^n.)

② Let $\eta \in \mathcal{B}_n$ be equiv. to a positive word β.

Then $\boxed{X(\eta\sigma_n)/V(\eta\sigma_n) \cong X(\eta)/V(\eta) \times \mathbb{C}^\times}$.

highly non-triv., need to show $\Lambda(u,w) \cong$ Leg. isot. $\Lambda(f)$

Corollary: $\mathring{\Pi}_{u,w} \cong X(R_n(u,w)\Delta_n)/V \cong X(J_k(f)) \times (\mathbb{C}^\times)^{n-k}$.

How is the pair $(X(\eta), V(\eta))$ built? ← construct a DG-algebra $\mathcal{A}$ & find a good model for $H^0(\mathcal{A})$.

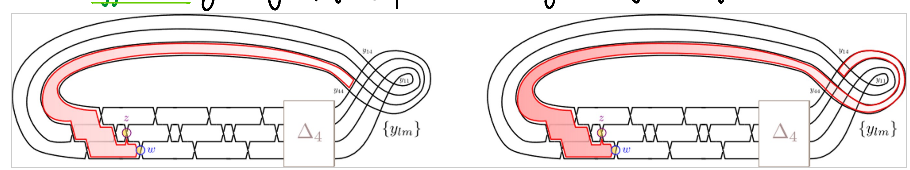
Let $\eta \in \mathcal{B}_n$ be equiv. to a positive word β. The DG-algebra $\mathcal{A}(\eta)$ is built as follows:

- freely graded commutative generated by $(\longrightarrow$ also $\mathbb{R}_{\geq 0}$-filtered$)$

DGA $\mathcal{A}(\eta)$

y_{ij} degree 1, $i,j = 1, \dots, n$; z_j degree 0, $j = 1, \dots, \ell^+(\eta) + \binom{n}{2}$ "positive crossings" of $\eta\Delta$; w_k degree -1, $k = 1, \dots, \ell^-(\eta)$ "negative crossings" of $\eta\Delta$

- differential given by $\mathcal{B}_i(z_j, w_k)$ products counting: $0 \xrightarrow{\partial} \langle y_{\ell m} \rangle \xrightarrow{\partial} \langle z_j \rangle \xrightarrow{\partial} \langle w_k \rangle \xrightarrow{\partial} 0$



Thm: Let $\eta \in \mathcal{B}_n$ be equiv. to a positive word β.

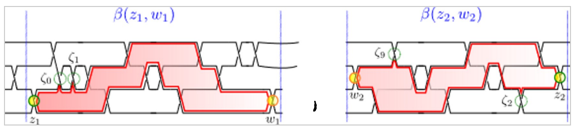
Then $\partial : A(\eta) \longrightarrow A(\eta)$ satisfies $\boxed{\partial^2 = 0}$.

and its cohomology $H^0(A(\eta))$ is invariant under braid moves & Δ-conjugations.

Moreover, $H^0(A(\eta))$ is given by the affine variety $X(\eta) = \{ z_s : \partial y_{ij} = 0 \}$ modulo $V(\eta) = \langle w_k^\vee \rangle$.

$\Longrightarrow$ the quotient $X(\eta)/V(\eta)$ is the desired invariant.

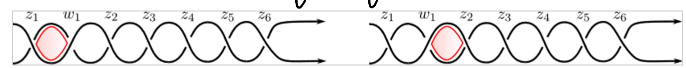
Example of a vector field contribution



Example: (Trefoil) We discussed $X(\sigma_1^4)$ already, with braid  σ_1^4 Δ,
this gave the open positroid $\Pi_{2,5} \subseteq Gr(2,5)$, and A_2-clusters.

$\rightarrow$ What if we want $X(\sigma_1 \cdot \sigma_1^{-1} \cdot \sigma_1^4)$ instead?

Then $\partial : \{z_j\} \longrightarrow \{w_k\}$ is given by



and $X(\eta) = \{ (z_1, \dots, z_6) \in \mathbb{C}^6 : B_1(z_1) B_1(0) B_1(z_2) \cdots B_1(z_6) + Id = 0 \} \subseteq \mathbb{C}^6$

with $V(\eta)$ generated by $t \cdot (z_1, \dots, z_6) \longrightarrow (z_1 + t, z_2 - t, z_3, \dots, z_6)$.

In fact, $B_1(z_1) B_1(0) B_1(z_2) \cdots B_1(z_6)$ is a function on $z_1 + z_2$, so quotient is direct. ∎

THE END

Thank you!