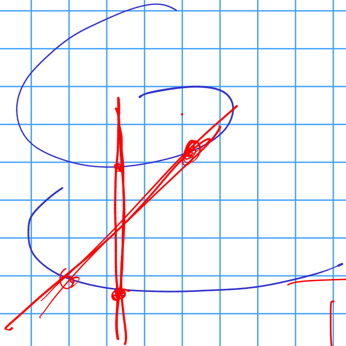
$X \subset \mathbb{P}^N$



FOR SMALL r $\quad K_r = \sigma_r$

$R \subset X$ WHICH ARE NOT "SMOOTHABLE"

$$K_r(X) \subset \mathbb{P}^N$$

$$\overbrace{\bigcup \left\{ \langle R \subset X \text{ 0-dim SUBSCH OF LENGTH } r \rangle \right\}}^{\text{"}}$$

Fujita vanishing, sufficiently ample line bundles, and cactus varieties

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Ample and very ample line bundles

X projective scheme (or variety) over $\Bbbk = \overline{\Bbbk}$,
L line bundle on X,
$\phi_L \colon X \dashrightarrow \mathbb{P}(H^0(L)^*)$ the map determined by the global sections of L:

$$\phi_L(x)(s) := [s(x)], \text{ where } x \in X \text{ and } s \in H^0(L).$$

Definition

We say L is very ample if $\phi_L \colon X \hookrightarrow \mathbb{P}\left(H^0(L)^*\right)$ (if it is an embedding) and $L \simeq \phi_L^* \mathcal{O}_{\mathbb{P}(H^0(L)^*)}(1)$.

Definition

We say L is ample if $L^{\otimes m}$ is very ample for some integer $m > 0$.

Things you probably know about (very) ampleness

1. L very ample, L' very ample $\Longrightarrow L \otimes L'$ very ample (diagonal and Segre embeddings):

$$\begin{tikzcd}
X \arrow[rr, dotted, hook, "\phi_{L\otimes L'}"] \arrow[d, hook, "\text{diagonal}"] & & \mathbb{P}\left(H^0(L \otimes L')^*\right) \arrow[d, dashed, "\text{linear}"] \\
X \times X \arrow[d, hook, "\phi_L \times \phi_{L'}"] & & \\
\mathbb{P}\left(H^0(L)^*\right) \times \mathbb{P}\left(H^0(L')^*\right) \arrow[rr, hook, "\text{Segre}"] & & \mathbb{P}\left(\left(H^0(L) \otimes H^0(L')\right)^*\right).
\end{tikzcd}$$

Things you probably know about (very) ampleness

1. L very ample, L' very ample $\Longrightarrow L \otimes L'$ very ample (diagonal and Segre embeddings).
2. L ample, L' ample $\Longrightarrow L \otimes L'$ ample (first item + definition of ample).
3. L very ample, L' ample $\Longrightarrow L \otimes L'$ ample (second item + very ample is ample).
4. L very ample, L' ample $\Longrightarrow L \otimes L'$ very ample ??

GENUS 3 CURVE

$$C = \{ x^4 + y^4 + yz^3 = 0 \} \subset \mathbb{P}^2$$

$P \in C \qquad P = [0, 0, 1]$

$(y = 0) \subset \mathbb{P}^2 \qquad C \cap (y=0) = 4P \subset C$

$4P$ – VERY AMPLE

P – AMPLE

$4P + P = 5P$ – NOT V. A.

Curve of genus 3

Example

Consider a smooth plane curve $C = \left\{x^4 + y^4 + yz^3 = 0\right\} \subset \mathbb{P}^2$ of genus 3, and let $P = [0, 0, 1] \in C$. Then the hyperplane section $y = 0$ is the divisor $4P$ on C, thus the corresponding line bundle $\mathcal{O}_C(4P)$ is very ample and $\mathcal{O}_C(P)$ is ample. However, $\mathcal{O}_C(5P)$ is not very ample! (It follows from Riemann-Roch that

$$H^0(\mathcal{O}_C(5P)) = H^0(\mathcal{O}_C(P)) \cdot H^0(\mathcal{O}_C(4P)),$$

thus every section of $\mathcal{O}_C(5P)$ vanishes at P.)

L	$L^{\otimes 2}$	$L^{\otimes 3}$	$L^{\otimes 4}$	$L^{\otimes 5}$	$L^{\otimes 6}$	$L^{\otimes 7}$	$L^{\otimes 8}$	$\dots$
NOT	V.A	V.A	NOT V.A	NOT V.A	V.A	V.A	V.A.	

High powers of ample bundles are very ample

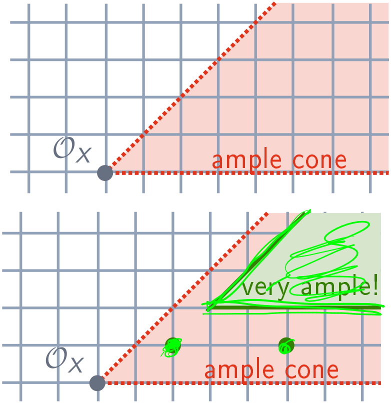
4. L very ample, L' ample, then $L \otimes L'$ is ample but not necessarily very ample.
5. L ample, then $L^{\otimes m}$ is very ample for all $m \gg 0$ (so it cannot happen that, say all even powers are very ample but all odd powers are not very ample). It follows from (a relative version) of Serre's vanishing theorem.

Sufficiently ample bundles are very ample

The group Pic(X) of line bundles on X can have "more dimensions" than just powers of a single line bundle.

Pic(X) :

$\mathbb{Z}^2$

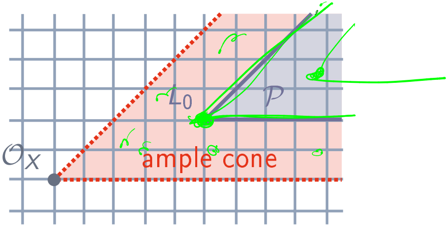


OPEN CONE IN Pic(X) $\otimes \mathbb{R}$

6. $\exists$ line bundle L such that $\forall$ ample L' the product $L \otimes L'$ is very ample.

Sufficiently ample

Pic(X) :



Definition

Let $\mathcal{P}$ be a property of line bundles on X. We say that sufficiently ample line bundle satisfies $\mathcal{P}$ if there exists a line bundle L_0 such that for any ample L the product $L_0 \otimes L$ satisfies $\mathcal{P}$. (Equivalently: for any L nef — see figure.)

Serre and Fujita vanishing

Let $\mathcal{F}$ be a coherent sheaf on X.

Theorem (Serre vanishing)

For any ample line bundle L and $m \gg 0$ and any $i > 0$ we have

$$H^i(\mathcal{F} \otimes L^{\otimes m}) = 0$$

Theorem (Fujita vanishing)

For a sufficiently ample line bundle L and any $i > 0$ we have

$$H^i(\mathcal{F} \otimes L) = 0.$$

That is, there exists L_0 such that for any L that is equal to L_0 + ample, the higher cohomologies of $\mathcal{F} \otimes L$ vanish.

There are also relative versions of the same theorems.

RELATIVE FUJITA VANISHING

X, Y PROJ VARIETIES $\mathcal{F}$ SHEAF ON $X \times Y$ FLAT OVER Y.

$\mathcal{F}$ A FAMILY OF SHEAVES ON X

FOR A SUFF AMPLE L ON X

AND ANY $y \in Y$ $H^i(\mathcal{F}_y \otimes L) = 0$ $i > 0$

RELATIVE FUJITA VANISHING IS USEFUL TO PROVE THAT SUFFICIENTLY AMPLE LINE BUNDLE IS VERY AMPLE!

$\mathcal{I}_{\mathcal{R}} \subset \mathcal{O}_{Hilb_2(X) \times X}$

$Hilb_2(X) = Y = \{$ deg 2 SUBSCHEMES OF X $\}$

$H^1(I_{\cdot\cdot} \otimes L) = 0$

$H^1(I_{\cdot\nearrow} \otimes L) = 0$!

$H^0 L \twoheadrightarrow L|_{:}$

$\twoheadrightarrow L|_{\cdot\nearrow}$

Examples of properties of sufficiently ample line bundles

Example

Sufficiently ample line bundle is very ample.

Example

The embedding $\phi_L(X) \subset \mathbb{P}\left(H^0(L)^*\right)$ by a sufficiently ample line bundle L is projectively normal.

X – NORMAL VARIETY

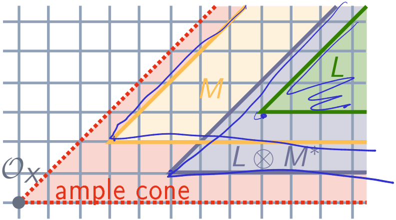
Theorem (Sidman-Smith, 2011)

The embedding $\phi_L(X) \subset \mathbb{P}\left(H^0(L)^*\right)$ by a sufficiently ample line bundle L has ideal generated by 2×2 minors of a matrix with linear entries.

Matrices with linear entries

Write: $L \simeq M \otimes (L \otimes M^*)$. We want both M and $(L \otimes M^*)$ to be sufficiently ample.

$\mathrm{Pic}(X)$:

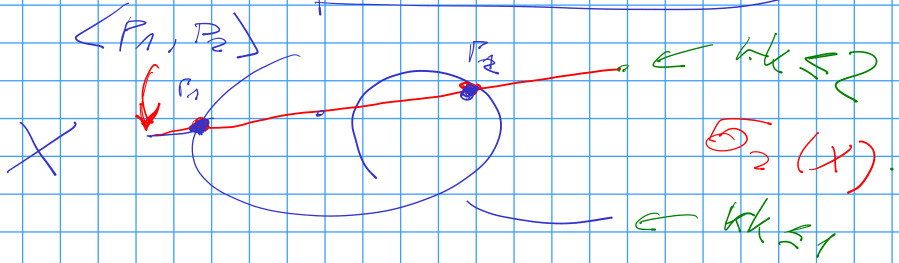


$$
\begin{tikzcd}
X \arrow[r, "\phi_L"] \arrow[d, hook, "\text{diagonal}"] & \mathbb{P}\left(H^0(L)^*\right) \arrow[dd, "\text{linear}"] \\
X \times X \arrow[d, hook, "\phi_M \times \phi_{L\otimes M^*}"] & \\
\mathbb{P}\left(H^0(M)^*\right) \times \mathbb{P}\left(H^0(L \otimes M^*)^*\right) \arrow[r, hook, "\text{Segre}"] & \mathbb{P}\left(H^0(M)^* \otimes H^0(L \otimes M^*)^*\right).
\end{tikzcd}
$$

$$L = M \otimes (M^* \otimes L)$$

$$((2 \times 2) \text{ minors}) = I$$

$$X = \left\{ rk \left(\overset{M}{\text{matrix with linear entries}} \right) \leq 1 \right\}$$



Secant varieties

CURVES '80 ~2011

Conjecture (Eisenbud-Koh-Stillman, Sidman-Smith)

For any fixed r, the r-th secant variety to $\phi_L(X)$ for a sufficiently ample line bundle L has ideal generated by $(r+1) \times (r+1)$ minors of a matrix with linear entries.

X – VARIETY

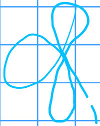
Definition

$X \subset \mathbb{P}^N$, then the r-th secant variety of X is:

$$\sigma_r(X) := \overline{\bigcup \{ \langle R \rangle \mid R \subset X \text{ finite smooth subscheme of length} \leqslant r \}}.$$

LINEAR SPAN

COUNTEREXAMPLES TO EKS & SS CONJECTURES:

1) IF X = SUFFICIENTLY SINGULAR CURVE
$\Rightarrow$ NO CHANCE FOR

CONJ. TO BE TRUE
$r = 2$ (OR HIGHER)

2) IF $X = \mathbb{P}^n$, THEN $n \geq 6$
CONJECTURE FAILS FOR
$r \gg 0$ $(r \geq 14)$

Cactus varieties

Conjecture (Corrected)

For any fixed r, the r-th ~~secant variety~~ cactus variety to $\phi_L(X)$ for a sufficiently ample line bundle L has ideal generated by $(r+1) \times (r+1)$ minors of a matrix with linear entries.

Definition

$X \subset \mathbb{P}^N$, then the r-th cactus variety of X is:

$\sigma_r(X)$ smooth

$\sigma_r(X) \subset \mathfrak{K}_r(X)$

$$\mathfrak{K}_r(X) := \overline{\bigcup \{\langle R \rangle \mid R \subset X \text{ finite subscheme of length} \leqslant r\}}. \subsetneq$$

$\mathfrak{K}_r(X) \subset \{rk \leq r\}$

X - smooth $r \geq 14$
$\dim X \geq 6$

Cactus varieties to sufficiently ample embeddings

Theorem (Buczyńska, B., Farnik)

X projective variety, $r > 0$ an interger. Then the r-th cactus variety of $\phi_L(X)$ is set-theoretically defined by $(r+1) \times (r+1)$ minors of a matrix with linear entries for any sufficiently ample line bundle L.

(Work in progress with Hanieh Keneshlou: we hope to have a scheme-theoretic or maybe even ideal-theoretic analogue.)

IDEA OF PROOF:

1) PROJECTIVE NORMALITY:

$$S_L = \bigoplus_{m \in \mathbb{Z}\geq 0} H^0(X, L^m)$$

X - (WEAKLY) P.N. $\Longleftrightarrow$ S_L IS GENERATED BY $H^0(X, L)$.

(IF X NORMAL W.P.N. $\Longleftrightarrow$ P.N.)

$\forall m \geq 1$ $H^0(X, L) \otimes H^0(X, L^{m-1}) \twoheadrightarrow H^0(L^m)$

2a) SUFFICIENTLY AMPLE BUNDLES ON PRODUCT.

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

$L_1 \boxtimes L_2 = \pi_1^* L_1 \otimes \pi_2^* L_2$

L_1, L_2 L.B ON X.

IF $\mathcal{P}$ PROPERTY OF L. B ON $X \times X$ WHICH IS SATISFIED BY SUFF. AMPLE L. B. ON $X \times X$.

$\Rightarrow$ IF L_1 & L_2 ARE SUFF. A. L.B ON X

THEN $L_1 \boxtimes L_2$ SATISFIES P.

$\mathrm{Pic}\, X \boxtimes \mathrm{Pic}\, X \subset \mathrm{Pic}\, X \times X$

$\cup$ OPEN $\qquad\qquad \cup$

$\mathrm{Amp}_{X \boxtimes X} \subset$ Amp. Too

26) DOUBLE PROJ NORMALITY

L_1, L_2 ON X

(L_1, L_2) IS DOUBLY P.N.

$$S_{L_1, L_2} = \bigoplus_{m,n} H^0(X, L_1^m \otimes L_2^n)$$

IS GENERATED IN DEGREE

$(1,0) \otimes (0,1)$

$H^0(X, L_1) \otimes H^0(X, L_2)$

FUJITA VANISHING FOR

$I_\Delta \subset \mathcal{O}_{X \times X}$

$\Rightarrow$ L_1, L_2 ARE SUFF. AMPLE $\Rightarrow$

S_{L_1, L_2} IS GENERATED BY $(1,0)$, $(0,1)$ GRADING.

($\Longrightarrow$ PR. NORMALITY)

3) CONSTRUCTING DOUBLE GRADED ALGEBRAS

$$\phi_L(X) \subset \mathbb{P}\left(H^0(L)^*\right)$$

$$K_r(\phi_L(X)) \subset \{p : rk(\cdot) \le r\}$$

$\subset$ EASIER PART

$\supseteq$ DIFF. PART

L_1, L_2 S.T. V.A. ,

DOUBLE PROJ. NORMAL.

$$L = L_1^{\otimes 4r-1} \otimes L_2$$

$$S_{L_1, L_2} \qquad H^0(L) = (S_{L_1 L_2})_{4r-1}$$

$$\mathrm{Pic}\, X \supset \overline{\mathrm{Ample}(X)}$$

The end

Thank you for attention!