The Integer Decomposition Property and Ehrhart Unimodality for Weighted Projective Space Simplices

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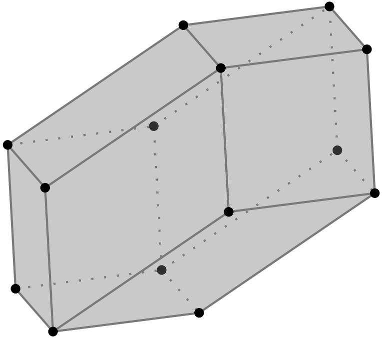
Lattice Polytopes

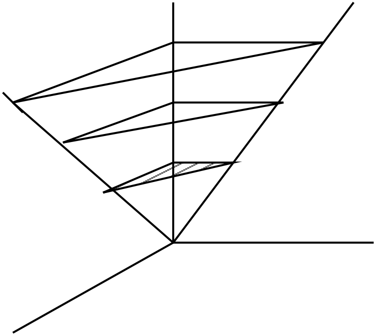
Let $P = \text{conv}\{v_1, \ldots, v_n\}$ with $v_i \in \mathbb{Z}^d$ be a d-dimensional lattice polytope in $\mathbb{R}^d$.



A lattice polytope.

The cone over P

$$\mathrm{cone}(P) = \mathsf{span}_{\mathbb{R}_{\geq 0}}\{(1, v_1), \ldots, (1, v_n)\} \subset \mathbb{R}^{d+1}$$



The cone over a triangle.

Arithmetic ∩ Geometry

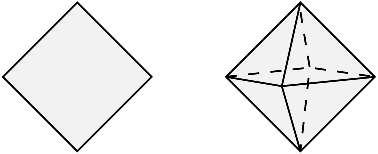
P has the *Integer Decomposition Property*, or P is IDP, if

- for every $(k, y) \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}$,
- there exist $(1, x_1), \ldots, (1, x_k) \in (1, P) \cap \mathbb{Z}^{d+1}$
- with $(1, x_1) + \cdots + (1, x_k) = (k, y)$.

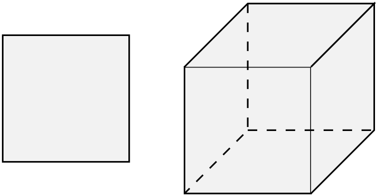
P is *reflexive* if the dual of P is also a lattice polytope.

Remark: IDP is equivalent to projective normality of the associated toric variety, reflexive implies Gorenstein Fano.

Some examples that many people like



Examples of lattice crosspolytopes...



with duals given by lattice $(-1/1)$-cubes. All of these are IDP.

General Challenges Regarding Lattice Polytopes

The following tasks are all quite difficult:

- Identifying when P is IDP.
- Classifying reflexive polytopes of fixed dimension.
- Classifying lattice polytopes of fixed dimension and bounded volume.
- Determining algebraic invariants (Ehrhart series, Poincaré series) associated to $\mathrm{cone}(P)$.

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Comment: There are a range of interesting conjectures about all (or large classes of) lattice polytopes motivated by well-behaved families of combinatorially-defined polytopes.

Main questions

We are motivated by the following questions (h^*-vectors will be defined later):

Question (T. Hibi)

Do all reflexive polytopes have unimodal h^*-vectors?

No. Counterexamples due to S. Payne and M. Mustaţă̆.

Question (special case of a conjecture due to F. Brenti, also asked by T. Hibi & H. Ohsugi)

Do all IDP reflexive polytopes have unimodal h^*-vectors?

Question (special case of a conjecture due to F. Brenti, also asked by J. Schepers & L. Van Langenhoven)

Do all IDP lattice polytopes have unimodal h^*-vectors?

The latter two questions are (very) open, and fall within a broader set of questions about reflexivity, IDP, very ampleness, and Ehrhart h^*-vectors.

Our project goals

- Focus on a specific family of lattice simplices that is "sufficiently complicated" in that they are farther away from the combinatorics, but which admit (A) reasonable computational experimentation in high dimensions and (B) concrete tools for proving properties like IDP and reflexive.
- First focus on investigating those simplices that are both reflexive and IDP, then turn attention to unimodality of h^*-vectors.

The family of $\Delta_{(1,q)}$ simplices

For $q = (q_1, q_2, \ldots, q_d) \in \mathbb{Z}^d_{\geq 1}$ and $e_i \in \mathbb{R}^d$ the i-th standard basis vector, define

$$\Delta_{(1,q)} := \mathrm{conv}(e_1, e_2, \ldots, e_d, -q)\,.$$

Motivation:

1. These correspond to certain weighted projective spaces.
2. These have a Hermite normal form that is "one-column", a class of simplices considered by Hibi, Higashitani, and Li.
3. In principle, all properties of $\Delta_{(1,q)}$ should be obtainable from number-theoretic properties of q.

$\Delta_{(1,q)}$ and reflexivity

Theorem (Conrads/folklore)

$\Delta_{(1,q)}$ is reflexive if and only if for all $j = 1, \dots, d$

$$q_j \text{ divides } \left(1 + \sum_i q_i\right),$$

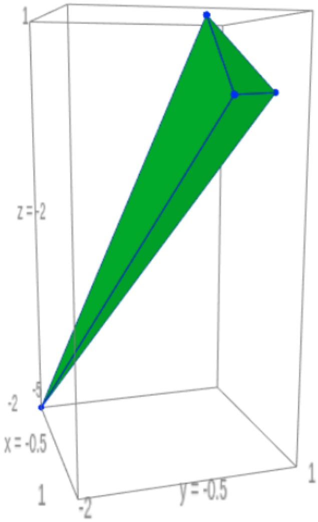
i.e. there exists some $k \in \mathbb{Z}_{>0}$ such that

$$1 + \sum_i q_i = k \cdot q_j .$$

The proof follows immediately from the duality definition of reflexivity.

The *normalized volume* of $\Delta_{(1,q)}$ is $n(q) := 1 + \sum_i q_i$.

$q = (2, 2, 5)$ has $1 + 2 + 2 + 5 = 10$



Specifying multiplicities of distinct weights in q

We say that both q and $\Delta_{(1,q)}$ are *supported* by $r = (r_1 < r_2 < \cdots < r_k)$ if there exist positive integers $x_1, \ldots, x_k$ such that

$$q = (q_1, \ldots, q_d) = (r_1^{x_1}, r_2^{x_2}, \ldots, r_k^{x_k}) = (r, x)$$

where $r_i^{x_i}$ indicates that r_i has multiplicity x_i.

Example

$(2, 2, 5) = (2^2, 5^1) = ((2, 5), (2, 1))$

Infinitely many reflexive IDP $\Delta_{(1,q)}$'s exist...

- For each r-vector, BB, R. Davis, and L. Solus (2018) proved there are infinitely many reflexive $\Delta_{(1,q)}$ supported on r.
- BB and R. Davis (2016) proved that given q and p such that $\Delta_{(1,q)}$ and $\Delta_{(1,p)}$ are both reflexive and IDP, a new vector $p * q$ can be constructed such that $\Delta_{(1,p*q)}$ is reflexive and IDP.
- This corresponds to the *affine free sum* construction for rational polytopes.
- Thus, there are infinitely many reflexive IDP $\Delta_{(1,q)}$'s that arise in this manner.
- We will soon see that for a fixed r having some $r_j \nmid r_k$, only finitely many reflexive $\Delta_{(1,q)}$'s supported on r are IDP.

yet arithmetically interesting IDP reflexive $\Delta_{(1,q)}$ seem rare

Consider all r-vectors that are partitions of $N \leq 75$ with distinct entries, such that there exists some r_j such that $r_j \nmid r_k$.

# of r-vectors of this type	# of IDP reflexives supported by them
501350	509

Note: In reality, this data provides a limited glimpse, e.g.,

$$q = (210, 211, 211, 211, 211, \underbrace{1055, 1055, \ldots, 1055}_{41 \text{ times}})$$

is not among this sample, but it is both IDP and reflexive with $210 \nmid 1055$.

When is a reflexive $\Delta_{(1,q)}$ also IDP?

Theorem (BB, R. Davis, L. Solus, 2018)

The reflexive simplex $\Delta_{(1,q)}$ is IDP if and only if for every $j = 1, \ldots, d$, for all $b = 0, 1, \ldots, q_j - 1$ satisfying

$$b\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i\neq j}\left\lfloor \frac{bq_i}{q_j}\right\rfloor \geq 2 \tag{1}$$

there exists a positive integer $c < b$ satisfying the following equations, where the first is considered for all $i \neq j$ between 1 and d:

$$\left\lfloor \frac{(b-c)q_i}{q_j}\right\rfloor + \left\lfloor \frac{cq_i}{q_j}\right\rfloor = \left\lfloor \frac{bq_i}{q_j}\right\rfloor, \text{ and} \tag{2}$$

$$c\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i\neq j}\left\lfloor \frac{cq_i}{q_j}\right\rfloor = 1. \tag{3}$$

A Useful Necessary Condition

We call the following the *necessary condition*.

Theorem (BB, R. Davis, L. Solus, 2018)

If q is reflexive and IDP, then for each $1 \leq j \leq d$,

$$1 + \sum_{i=1}^{d} (q_i \bmod q_j) = q_j \, .$$

Note: Compare this to $\Delta_{(1,q)}$ reflexive if and only if

$$1 + \sum_{i=1}^{d} q_i = k \cdot q_j \, .$$

Reframing The Necessary Condition

Theorem (BB, R. Davis, L. Solus, 2018)

If $q = (r, x)$ is reflexive and IDP, then for each $1 \leq j \leq k$,

$$1 + \sum_{i=1}^{k} x_i (r_i \bmod r_j) = r_j \, .$$

The following corollary is very useful.

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The following corollary is very useful.

Corollary (BB, R. Davis, M. Lane, L. Solus, 2020+)

If $q = (r, x)$ has $\Delta_{(1,q)}$ reflexive and IDP, and if there exists some r_j such that $r_j \nmid r_k$, then

- $1 \leq x_i \leq r_{i+1}/r_i$
- $1 \leq x_k \leq r_j/(r_k \bmod r_j)$

A change in perspective

The previous corollary is a restriction on the multiplicities in terms of the entries.

We should also consider what happens when we think of the entries in terms of the multiplicities.

1- and 2-supported $\Delta_{(1,q)}$

Proposition

If $q = (r_1^{x_1})$ is IDP reflexive, then $q = (1, 1, \ldots, 1)$.

Theorem (BB, R. Davis, L. Solus 2018)

For the vector $q = (r_1^{x_1}, r_2^{x_2})$, $\Delta_{(1,q)}$ is IDP reflexive if and only if it satisfies the necessary condition. The following is a classification of all such vectors, for $x_1, x_2 \geq 1$:

- $q = (1^{x_1}, (1 + x_1)^{x_2})$
- $q = ((1 + x_2)^{x_1}, (1 + (1 + x_2)x_1)^{x_2})$

Remark: In the first case, $r_1 \mid r_2$. In the second, $r_1 \nmid r_2$.

3-supported $\Delta_{(1,q)}$

Theorem (BB, R. Davis, M. Lane, L. Solus, 2020+)

Consider a 3-supported vector $q = (r, x)$ such that $\Delta_{(1,q)}$ satisfies the necessity condition. If $x = (x_1, x_2, x_3)$ is the multiplicity vector, then r is of one of the following eight forms (see next slide):

1. $(1, 1 + x_1, (1 + x_1)(1 + x_2))$.
2. $(1 + x_2, 1 + x_1(1 + x_2), (1 + x_1(1 + x_2))(1 + x_2))$.
3. $((1 + x_2)(1 + x_3), 1 + x_1(1 + x_2)(1 + x_3), (1 + x_1(1 + x_2)(1 + x_3))(1 + x_2))$.
4. $(1, (1 + x_1)(1 + x_3), (1 + x_1)(1 + x_2(1 + x_3)))$.
5. $(1 + (1 + x_3)x_2, (1 + x_3)(1 + x_1(1 + (1 + x_3)x_2)), (1 + (1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2))$.
6. $((1 + x_3)(1 + (1 + x_3)x_2), (1 + x_3)(1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)), (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2))$.
7. $(1+x_3, (1+x_3)(1+x_1(1+x_3)), (1+(1+x_3)x_1)(1+(1+x_3)x_2))$.

8. There exists some $k, s \geq 1$, where $r = (1 + kx_2, (skx_2 + s + k)(1 + x_1(1 + kx_2)), (1 + x_1(1 + kx_2))(1 + x_2(skx_2 + s + k)))$ and $x = (skx_2 + s + k - 1, x_2, x_3)$.

Note that the first seven r-vectors each correspond to a unique divisibility criteria for $r = (r_1, r_2, r_3)$, as follows:

1. $r_1 \mid r_2, r_1 \mid r_3, r_2 \mid r_3$
2. $r_1 \nmid r_2, r_1 \mid r_3, r_2 \mid r_3$
3. $r_1 \nmid r_2, r_1 \nmid r_3, r_2 \mid r_3$
4. $r_1 \mid r_2, r_1 \mid r_3, r_2 \nmid r_3$
5. $r_1 \nmid r_2, r_1 \mid r_3, r_2 \nmid r_3$
6. $r_1 \nmid r_2, r_1 \nmid r_3, r_2 \nmid r_3$
7. $r_1 \mid r_2, r_1 \nmid r_3, r_2 \nmid r_3$
8. $r_1 \nmid r_2, r_1 \mid r_3, r_2 \nmid r_3$

Note that (5) and (8) share the same divisibility pattern. Of these families, only (8) appears to be non-IDP, and we are in the process of proving this.

The semigroup algebra associated to P

$\mathbb{C}[P]$ is the $\mathbb{C}$-semigroup algebra for $\mathrm{cone}(P) \cap \mathbb{Z}^{d+1}$, graded by the "height" coordinate.

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The *Hilbert/Ehrhart series* of P is

$$\sum_{t \geq 0} |tP \cap \mathbb{Z}^d| z^t = \sum_{t \geq 0} \dim_{\mathbb{C}}(\mathbb{C}[P]_t) z^t = \frac{\sum_{j=0}^{d} h_j^* z^j}{(1-z)^{d+1}}$$

where by a result due to Stanley it is known $h_j^* \in \mathbb{Z}_{\geq 0}$ for all j.

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where by a result due to Stanley it is known $h_j^* \in \mathbb{Z}_{\geq 0}$ for all j.

We call $h^*(P;z) = \sum_{j=0}^{d} h_j^* z^j$ the *h-star-polynomial* of P, with coefficient vector the h^**-vector* of P.

Symmetry and unimodality

Theorem (Hibi)

P is reflexive if and only if $h^*(P;z)$ has degree $d = \dim(P)$ and the h^*-vector of P is symmetric.

We say P is *h^*-unimodal* if the h^*-vector of P is unimodal.

A Deep Result

Theorem (Bruns and Römer)

If P is Gorenstein and admits a regular unimodular triangulation, then P is h^*-unimodal.

Remark: Reflexive implies Gorenstein, and regular unimodular triangulation implies IDP.

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Aside: The reflexive polytopes that are IDP reflexive but do not admit a regular unimodular triangulation are completely mysterious, both in general and for the $\Delta_{(1,q)}$ simplices.

Aside: In new work with my student Derek Hanely, we are investigating regular unimodular triangulations for 2-supported IDP reflexive $\Delta_{(1,q)}$'s.

$h^*(\Delta_{(1,q)}; z)$

Theorem (BB, R. Davis, L. Solus, 2018)

The h^*-polynomial of $\Delta_{(1,q)}$ is

$$\sum_{b=0}^{q_1+\cdots+q_d} z^{w(b)}$$

where

$$w(b) := b - \sum_{i=1}^{d} \left\lfloor \frac{q_i b}{1 + q_1 + \cdots + q_d} \right\rfloor .$$

What does this look like?

For $q = (3, 20, 24, 24, 24, 24) = ((3, 20, 24), (1, 1, 4))$:

- link to animation
- $h^* = [1, 16, 29, 28, 29, 16, 1]$
- The Hilbert basis for $\text{cone}(\Delta_{(1,q)})$ has only one element outside $(1, \Delta_{(1,q)})$, at height 2.

For $q = (1, 1, 1, 1, 1, 1, 1, 12, 40, 60) = ((1, 12, 40, 60), (7, 1, 1, 1))$:

- link to animation
- $h^* = [1, 7, 15, 14, 16, 14, 16, 14, 15, 7, 1]$
- The Hilbert basis for $\text{cone}(\Delta_{(1,q)})$ has only two elements outside $(1, \Delta_{(1,q)})$, both at height 2.

Naive Question

Ignoring IDP, reflexive, etc, do we expect $\Delta_{(1,q)}$ to be h^*-unimodal?

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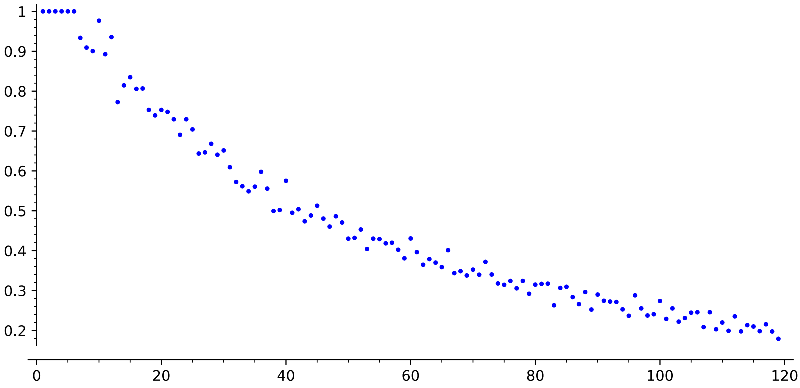
Let's experiment!

Partition the $\Delta_{(1,q)}$'s by $n(q) = 1 + \sum_i q_i$.

We have computed $h^*(\Delta_{(1,q)}; z)$ for **all** 61 537 394 $\Delta_{(1,q)}$ with $1 \leq n(q) \leq 75$, and randomly sampled q-vectors up to $n(q) = 120$.

Fraction of h^*-unimodal $\Delta_{(1,q)}$ with fixed normalized volume — based on random sampling



h^*-unimodality for $\Delta_{(1,q)}$ with small support

Theorem (BB, R. Davis, L. Solus, 2018)

If $\Delta_{(1,q)}$ is 2-supported and IDP reflexive, then $\Delta_{(1,q)}$ is h^*-unimodal.

Theorem (BB, R. Davis, M. Lane, L. Solus, 2020+)

If $\Delta_{(1,q)}$ is 3-supported and IDP reflexive, supported by r satisfying one of the following, then it is h^*-unimodal.

1. $r_1 \mid r_2, r_1 \mid r_3, r_2 \mid r_3$
2. $r_1 \nmid r_2, r_1 \mid r_3, r_2 \mid r_3$
3. $r_1 \nmid r_2, r_1 \nmid r_3, r_2 \mid r_3$
4. $r_1 \mid r_2, r_1 \mid r_3, r_2 \nmid r_3$

Remark: We have lots of computational evidence that for divisibility criteria 5–7, h^*-unimodality also holds. Unfortunately, the proofs of unimodality are not particularly enlightening.

Motivating question

What happens when we have a reflexive $\Delta_{(1,q)}$ with a very large multiplicity for one of the weights?

We begin by investigating this for the weight 1.

Reflexive stabilizations

Definition

Let $q \in \mathbb{Z}^n_{\geq 1}$. The *first reflexive stabilization* of q, denoted $\mathrm{rs}(q)$, is the vector $(1, 1, \ldots, 1, q)$ such that $\Delta_{(1,\mathrm{rs}(q))}$ is reflexive and the number of 1's prepended to q is the minimum necessary for this condition to hold; if q is reflexive, then no 1's are prepended.

We say the number of 1's prepended to q in $\mathrm{rs}(q)$ is the *reflexive stabilization number* of q, denoted $\mathrm{rsn}(q)$. Thus, we can write

$$\mathrm{rs}(q) := (1^{\mathrm{rsn}(q)}, q)\,.$$

Note that when prepending $\mathrm{lcm}\,(q)$ copies of 1 as many times as desired, the corresponding simplex will remain reflexive.

Higher reflexive stabilizations

More generally, the *m-th reflexive stabilization* of q, denoted $\mathrm{rs}(q, m)$, is defined as

$$\mathrm{rs}(q, m) := (1^{\mathrm{rsn}(q)+(m-1)\cdot\mathrm{lcm}(q)}, q)\,.$$

Example

Let $q = (2, 2, 3)$. Then $1 + 2 + 2 + 3 = 8$, and thus $\mathrm{rs}(q) = (1, 1, 1, 1, 2, 2, 3)$ is reflexive with $\mathrm{rsn}(q) = 4$.

Further, $\mathrm{rs}(q, 3) = (r, x) = ((1, 2, 3), (16, 2, 1))$.

Properties of reflexive stabilizations

Theorem (BB, R. Davis, M. Lane, L. Solus, 2020+)

Assume that $q \in \mathbb{Z}^d_{\geq 2}$*. For* $m \geq 2$*,* $\Delta_{(1,\mathrm{rs}(q,m))}$ *is not IDP.*

Theorem (BB, R. Davis, M. Lane, L. Solus, 2020+)

Assume that $q \in \mathbb{Z}^d_{\geq 2}$*. For* m *sufficiently large,* $\Delta_{(1,\mathrm{rs}(q,m))}$ *is not* h^**-unimodal. Further, the* h^**-vector of* $\Delta_{(1,\mathrm{rs}(q,m))}$ *contains only* 1*'s and* 2*'s.*

Key tool

Given a reflexive $q = (r, x)$, set $1 + \sum q_i = \ell \cdot \operatorname{lcm}(r_1, \ldots, r_k)$.

Let $s_i := \operatorname{lcm}(r_1, \ldots, r_k) / r_i$ for each $1 \leq i \leq k$ We define

$$g_r^x(z) := \sum_{0 \leq \alpha < \operatorname{lcm}(r_1, \ldots, r_d)} z^{u(\alpha)}$$

where

$$u(\alpha) = u_r^x(\alpha) := \alpha\ell - \sum_{i=1}^{d} x_i \left\lfloor \frac{\alpha}{s_i} \right\rfloor .$$

Key Tool

Theorem (Braun and Liu)

Given $q = (r, x)$*, if* $1 + \sum q_i = \ell \cdot \mathrm{lcm}\,(r_1, \ldots, r_k)$*, we have that*

$$h^*(\Delta_{(1,q)}; z) = \left(\sum_{t=0}^{\ell-1} z^t \right) \cdot g_r^x(z)\,.$$

Example

For $q = (1^7, 3^4, 5^5)$, we have

$$(z^2 + z + 1)(z^{14} + z^{11} + 2z^{10} + 2z^8 + 3z^7 + 2z^6 + 2z^4 + z^3 + 1)\,.$$

Note that in this case, $\ell = 3$ and a factor of $z^2 + z + 1$ appears in the h^*-polynomial.

Study of h^*-vectors for $\Delta_{(1,q)}$ is ongoing

- The wide range of arithmetic, geometric and enumerative properties for the $\Delta_{(1,q)}$ simplices demonstrate that they are "sufficiently complicated" to be interesting.
- One interesting aspect of these simplices is that their properties are number-theoretic in nature rather than combinatorial in nature (such as lattice polytopes arising from graphs, posets, matroids, etc).

Thank you!