Hochschild cohomology of partial flag varieties and Fano 3-folds

G/P : with Maxim Smirnov, arXiv: 1911.09414

Fano 3-folds : with Enrico Fatighenti and Fabio Tanturri, work-in-progress

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1. Hochschild cohomology

fix k field (of characteristic 0)

1945: Hochschild, for associative algebra A

$$\boxed{HH^n(A) := \operatorname{Ext}^n_{A\otimes A^{op}}(A, A)}$$

k-vector space

A A-bimodule $= A\otimes A^{op}$ module

1962: Hochschild–Kostant–Rosenberg: geometric description

for commutative + regular

$$\boxed{HH^n(A) \cong \Lambda^n T_{A/k}}$$

Hochschild homology:

$$HH_n(A) := \operatorname{Tor}_n^{A\otimes A^{op}}(A, A)$$

$$HH_n(A) \cong \Omega^n_{A/k}$$

1963: Gerstenhaber: rich algebraic structure $HH^\bullet(A) := \bigoplus_{n\geq 0} HH^n(A)$

graded-commutative ~~associative~~ product of degree 0

Lie bracket of degree -1

every $HH^n(A)$ is representation

$\Rightarrow [HH^1(A), HH^1(A)] \subset HH^1(A)$: Lie subalgebra

+ compatibility $\Rightarrow$ $\boxed{\text{Gerstenhaber algebra}}$

Deformation theory $\alpha \in HH^2(A)$ classifies first-order deformations

s.t. self-bracket $[\alpha,\alpha] \in$ in $HH^3(A)$ measures obstruction

deformations as associative algebra

1980s Gerstenhaber–Schack, $HH^\bullet$ for quasiprojective varieties.

* $$\boxed{HH^n(X) := \operatorname{Ext}^n_{X\times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)}$$

(Kontsevich definition, not Gerstenhaber–Schack)

* $$\boxed{HH^n(X) \cong \bigoplus_{p+q=n} H^p(X, \Lambda^q T_X)}$$

Hochschild–Kostant–Rosenberg

* algebraic structure on $HH^\bullet(X)$ (details...) in sheaf cohomology

polyvector fields: cup product, Schouten bracket — bigraded even

$\boxed{?}$ Kontsevich: need a fancy isomorphism

Deformation theory HH^2 = first-order deformations of cat X (Lowen–Van den Bergh)

$$HH^2(X) \overset{HKR}{\cong} \cancel{H^2(X,\mathcal{O}_X)} \oplus H^1(X,T_X) \oplus \boxed{H^0(X,\Lambda^2 T_X)}$$

$H^2(X,\mathcal{O}_X)$ = gerby deformations

$H^1(X,T_X)$: Kodaira–Spencer = geometric deformations

$H^0(X,\Lambda^2 T_X)$: pre-Poisson structures = noncommutative deformations

+ derived invariant, + Gerstenhaber calculus, + functoriality ...

→ encode these efficiently

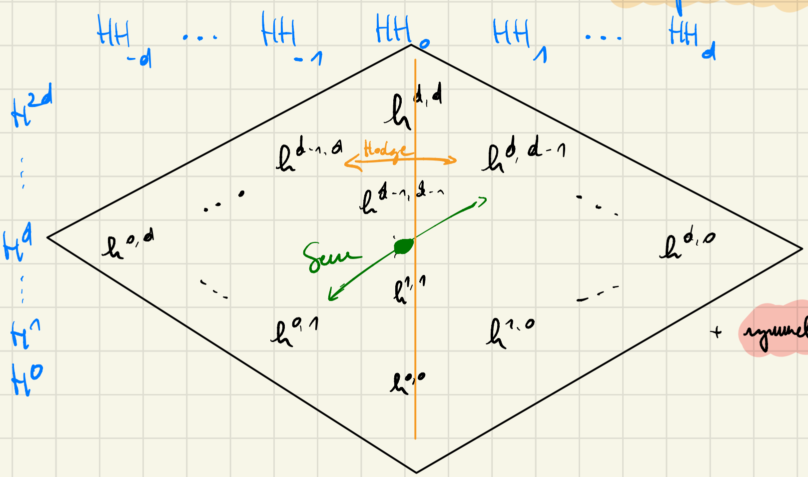
Today's question: can we determine $\boxed{H^p(X,\Lambda^q T_X)}$?

+ algebraic structure (or at least a part)

Hochschild homology $HH_n(X) \overset{HKR}{\cong} \bigoplus_{n=p-q} H^q(X,\Omega^p_X)$

Hodge decomposition: $H^n(X,\mathbb{C}) \cong \bigoplus_{n=p+q} H^q(X,\Omega^p_X)$ $h^{p,q} = \dim$

dimensions $h^{p,q}$ are collected in Hodge diamond

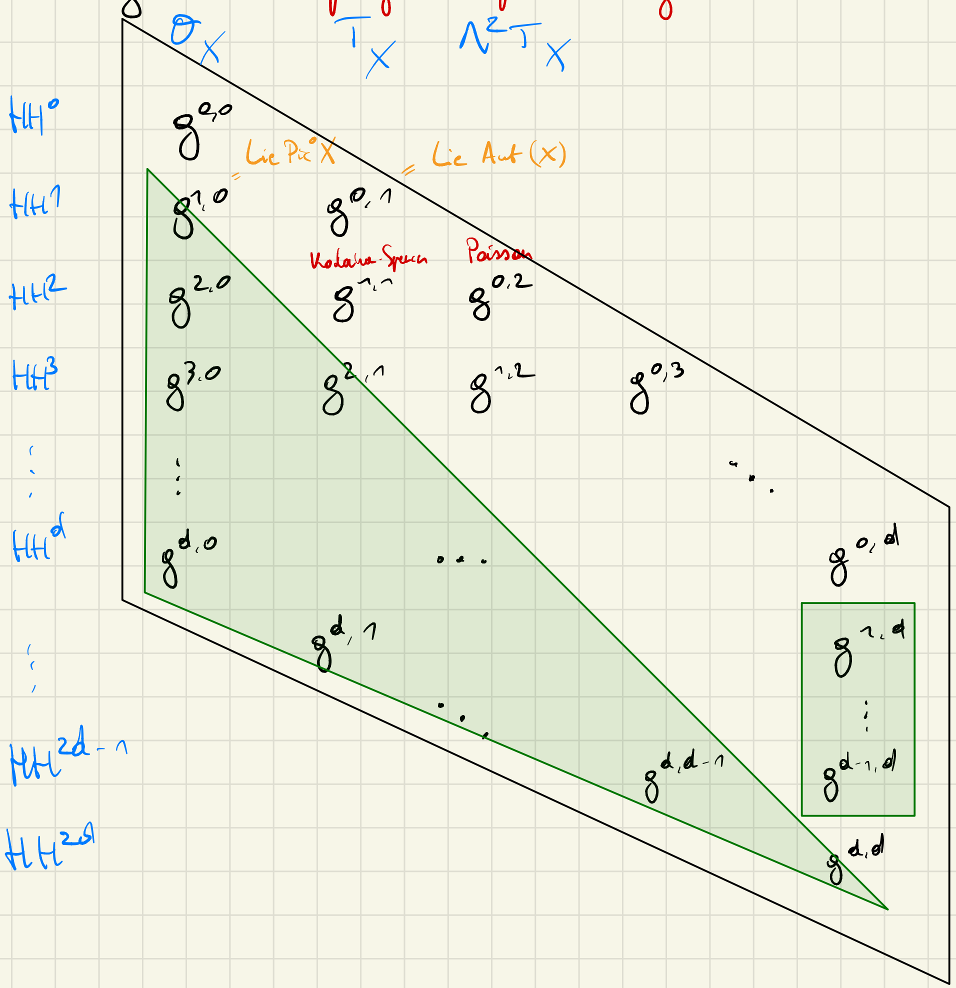


+ symmetries: Hodge, Serre

= familiar, ∃ industry to compute Hodge numbers

Hochschild cohomology $g^{p,q} := \dim H^p(X, \Lambda^q T_X)$

organise in polyvector parallelogram



= less familiar, $\nexists$ symmetries + link to deformation theory

X Fano: roughly half = lower diagonal vanishes
= Kodaira–Nakano

vanishing

⇒ CHALLENGE

2. Partial flag varieties

Setup $\quad G \quad$ simple reductive algebraic group $\qquad GL_n$

$\cup$

$P \quad$ parabolic subgroup $\qquad \begin{array}{c} k \\ n-k \end{array} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$\cup$

$B \quad$ Borel subgroup $\qquad$ upper triangular

G/B projective

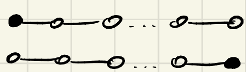
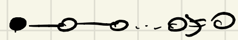
$\Rightarrow \quad G/P \quad$ smooth projective Fano variety

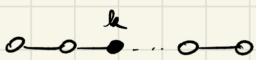
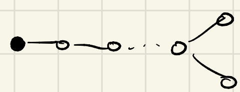
Idea: use representation theory of G and P to describe invariants of G/P

focus on singletons = maximal parabolic = generalized Grassmannians

Classification of G/Ps

$\{ G/P \} \quad \longleftrightarrow \quad$ Dynkin diagrams + subsets of vertices

e.g. $\mathbb{P}^{n-1}$ via  A_n $\qquad B_n$  gives Q^{2n-1}

$Gr(k,n)$ via  A_n $\qquad D_n$  gives Q^{2n-2}

Hochschild homology $\quad$ Hodge numbers via Borel – Hirzebruch, 1956

= start of using rep theory to do geometry

1) $h^{p,q} = 0$ if $p \neq q$

2) $h^{p,p} =$ via elements of length p in W/W_P

Hochschild affine
$\|$

Hochschild cohomology

* folklore (pre 2019) : $\boxed{H^p(X, \Lambda^q T_X) = 0}$ $\forall p \geq 1$

* evidence : OK for $\mathrm{Gr}(k,n)$, Q^n

* parallel : $H^p(X, \mathrm{Sym}^q T_X) = 0$ $\forall p \geq 1$, $\forall q \geq 0$

equiv. vector bundles

Problem : T_X, $\Lambda^q T_X$ is not nec. completely reducible

completely reducible

Borel-Weil-Bott : $H^\bullet(G/P, \mathcal{E}^\lambda)$

for λ highest weight of $L \subset P$
Levi

$\mathrm{coh}^G\, G/P \simeq \mathrm{rep}\, P$ not semisimple $T_X, \Lambda^q T_X$

$\cup$

$\mathrm{rep}\, L$ semisimple

NOT NECESSARILY APPLICABLE!

Vanishing theorem (implicit in Kono '87)

If G/P cominuscule or (co)adjoint

then $HH^\bullet(G/P)$ Hochschild affine

Description (B-Smirnov) for cominuscule or adjoint

$HH^i(G/P) = H^0(G/P, \Lambda^q T_{G/P})$ as $HH^1(G/P)$-representation

$= \mathfrak{g}$ Lie algebra of G

cominuscule

adjoint (the same for A, D, E)

adjoint for G/D

coadjoint ⇒ good properties

A_1	$\mathbb{P}^1$							
A_2	$\mathbb{P}^2$	$\mathbb{P}^{2,\vee}$						
A_3	$\mathbb{P}^3$	Gr(2,4)	$\mathbb{P}^{3,\vee}$					
A_4	$\mathbb{P}^4$	Gr(2,5)	Gr(3,5)	$\mathbb{P}^{4,\vee}$				
A_5	$\mathbb{P}^5$	Gr(2,6)	Gr(3,6)	Gr(4,6)	$\mathbb{P}^{5,\vee}$			
A_6	$\mathbb{P}^6$	Gr(2,7)	Gr(3,7)	Gr(4,7)	Gr(5,7)	$\mathbb{P}^{6,\vee}$		
A_7	$\mathbb{P}^7$	Gr(2,8)	Gr(3,8)	Gr(4,8)	Gr(5,8)	Gr(6,8)	$\mathbb{P}^{7,\vee}$	
C_2	$\mathbb{P}^3$	Q^3						
C_3	$\mathbb{P}^5$	SGr(2,6)	LGr(3,6)					
C_4	$\mathbb{P}^7$	SGr(2,8)	SGr(3,8)	LGr(4,8)				
C_5	$\mathbb{P}^9$	SGr(2,10)	SGr(3,10)	SGr(4,10)	LGr(5,10)			
C_6	$\mathbb{P}^{11}$	SGr(2,12)	SGr(3,12)	SGr(4,12)	SGr(5,12)	LGr(6,12)		
C_7	$\mathbb{P}^{13}$	SGr(2,14)	SGr(3,14)	SGr(4,14)	SGr(5,14)	SGr(6,14)	LGr(7,14)	
E_6	$\mathbb{OP}^2$	E_6/P_2	E_6/P_3	E_6/P_4	E_6/P_5	$\mathbb{OP}^{2,\vee}$		
E_7	E_7/P_1	E_7/P_2	E_7/P_3	E_7/P_4	E_7/P_5	E_7/P_6	E_7/P_7	
E_8	E_8/P_1	E_8/P_2	E_8/P_3	E_8/P_4	E_8/P_5	E_8/P_6	E_8/P_7	E_8/P_8
F_4	F_4/P_1	F_4/P_2	F_4/P_3	F_4/P_4				
G_2	Q^5	G_2/P_2						

B_2	Q^3	$\mathbb{P}^3$					
B_3	Q^5	OGr(2,7)	OGr(3,7)				
B_4	Q^7	OGr(2,9)	OGr(3,9)	OGr(4,9)			
B_5	Q^9	OGr(2,11)	OGr(3,11)	OGr(4,11)	OGr(5,11)		
B_6	Q^{11}	OGr(2,13)	OGr(3,13)	OGr(4,13)	OGr(5,13)	OGr(6,13)	
B_7	Q^{13}	OGr(2,15)	OGr(3,15)	OGr(4,15)	OGr(5,15)	OGr(6,15)	OGr(7,15)
D_3	Q^4	$\mathbb{P}^3$	$\mathbb{P}^3$				
D_4	Q^6	OGr(2,8)	Q^6	Q^6			
D_5	Q^8	OGr(2,10)	OGr(3,10)	$OGr_+(5,10)$	$OGr_+(5,10)$		
D_6	Q^{10}	OGr(2,12)	OGr(3,12)	OGr(4,12)	$OGr_+(6,12)$	$OGr_+(6,12)$	
D_7	Q^{12}	OGr(2,14)	OGr(3,14)	OGr(4,14)	OGr(5,14)	$OGr_+(7,14)$	$OGr_+(7,14)$

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E_8	E_8/P_1	E_8/P_2	E_8/P_3	E_8/P_4	E_8/P_5	E_8/P_6	E_8/P_7	E_8/P_8
F_4	F_4/P_1	F_4/P_2	F_4/P_3	F_4/P_4				
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D_4	Q^6	OGr(2,8)	Q^6	Q^6			
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E_7	E_7/P_1	E_7/P_2	E_7/P_3	E_7/P_4	E_7/P_5	E_7/P_6	E_7/P_7	
E_8	E_8/P_1	E_8/P_2	E_8/P_3	E_8/P_4	E_8/P_5	E_8/P_6	E_8/P_7	E_8/P_8
F_4	F_4/P_1	F_4/P_2	F_4/P_3	F_4/P_4				
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D_4	Q^6	OGr(2,8)	Q^6	Q^6			
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For coadjoint: no good description yet

<u>Non-vanishing</u> = folklore was wrong!

in fact, maximally wrong (?)

<u>Conjecture</u> if P maximal, G/P NOT cominuscule / (co)adjoint

then $HH^\bullet(G/P)$ <u>not</u> Hochschild affine

- lots of computational evidence: up to rank 10, except E_8
- explicit case: C_n/P_3 $\forall n \geq 4$

3. Fano 3-folds

1

0 dim Aut

0 dim $H^1(T_X)$ dim $H^0(\omega_X^\vee)$

0 0

0

for del Pezzo surfaces: pleasant exercise

how good is our understanding

for Fano 3-folds: * good test for our methods

* classification of Poisson structures

Hochschild homology

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \rho & & 0 & \\ 0 & & h^{2,1} & & h^{1,2} & & 0 \\ & 0 & & \rho & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

rk Pic(X)

$\boxed{?}$ $h^{p,q}$ are constant in family

$g^{p,q}$ are not nec. constant

follows from classification

used to distinguish different families

Poisson structures

$HH^2(X)$

Schouten

$$\alpha \in H^0(X, \Lambda^2 T_X) \text{ s.t. } [\alpha, \alpha] = 0 \text{ in } H^0(X, \Lambda^3 T_X)$$

Poisson surfaces: S s.t. $H^0(S, \omega_S^\vee) \neq 0$

dim 2: Bartocci – Macrì, 2004 vanishing for free

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dim 3 Fano, $\rho = 1$: Loray – Pereira – Touzet 2011

the $[\alpha, \alpha] = 0$ = strong condition

$H^0(X, \Lambda^2 T_X) \ni \alpha$

$[\alpha\,\alpha] \in H^0(X, \omega_X^\vee)$

$\shortparallel$

0

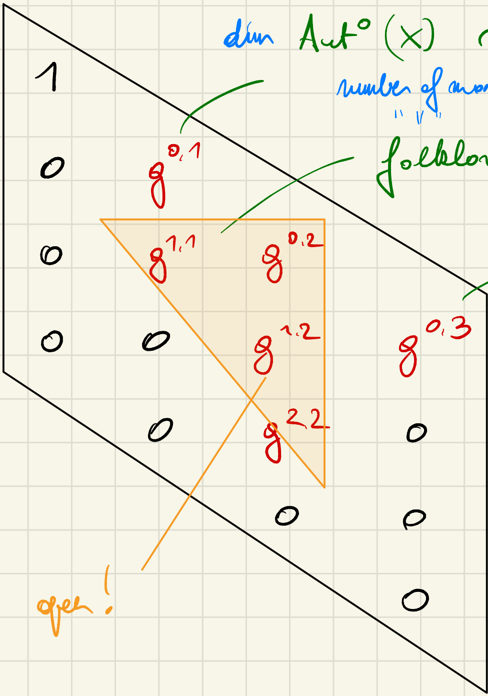
$\rho = 1$ 10th element in classification

e.g. $1\text{-}10 = \Lambda^3 \mathcal{U}^\vee$-section of $\mathrm{Gr}(3,7)$

$\exists$ jumping for dim Aut(X)

6-dim. family, $\exists!$ with $\exists!$ nonzero α up to rescaling s.t. $[\alpha\,\alpha] = 0$

$\dim H^0(X, \Lambda^2 T_X) = 3$ here



dim $\mathrm{Aut}^\circ(X)$ via Cheltsov–Przyjalkowski–Shramov, 2019

number of moduli

" "

folklore, ∄ proof

from classification by Mori–Mukai

open!

description of Fano 3-folds

1) Mori–Mukai: 1980s — birational

2) Coates–Corti–Galkin–Kasprzyk 2013: complete intersection in toric variety (+ a few others)

3) De Biase–Fatighenti–Tanturri 2020: zero loci equiv. bundles in (weighted) Grassmannians

(2) and (3) amenable to computer algebra — Koszul / equiv. bundles — toric / Borel–Weil–Bott

$\Rightarrow$ we know the missing numbers except for 2-1, 2-3, 4-13

Let's not look at 105 examples...

Conclusions: * "number of moduli" for Fano 3-folds

okin $e=1$ virtual dimension $= g^{1,1} - g^{0,1}$

* Poisson or blowup: $\exists$ recipe by Polishchuk Poisson

$\Rightarrow$ focus on primitive rk ≥ 2 for classification of Fano 3-folds

$H^0(X, \Lambda^2 T_X) = 0$ for 2-2, 2-6, 3-1

$\Rightarrow$ no Poisson structures! Also: need $[\alpha,\alpha]=0$

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lots of info, soon also this