UNDERSTANDING THE FLOP-FLOP AUTOEQUIVALENCE USING SPHERICAL FUNCTORS

CONVENTION:

All functors are implicitly derived.

The locus contracted by f_- has codimension ≥ 2

$+$

f_- contracts to a point an extremal curve C s.t. $K_{X_-} \cdot C < 0$.

Def. (What is a flop)

Given $f_- : X_- \longrightarrow Y$ a small extremal contraction s.t. ω_{X_-} is f_--trivial, a flop of f_- is a birational morphism $f_+ : X_+ \longrightarrow Y$ s.t.
(i) X_+ is not isom. to X_-
(ii) ω_{X_+} is f_+-trivial
(iii) X_+ has mild (terminal) singularities.

CONJECTURE: (BONDAL-ORLOV) In the situation above, we have $D^b(X_-) \simeq D^b(X_+)$.

$D^b(-) := D^b(Coh(-))$

EXAMPLE: ATIYAH FLOP

Fibre product

$$\begin{tikzcd}
 & Tot(\mathcal{O}(-1,-1)_{\mathbb{P}^1\times\mathbb{P}^1}) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X_- = Tot(\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1}) \arrow[dr] & & Tot(\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1}) = X_+ \arrow[dl] \\
 & Y = Spec\left(\mathbb{C}[a,b,c,d]/(ad-bc) \right) &
\end{tikzcd}$$

$\Rightarrow$ $D^b(X_-) \simeq D^b(X_+)$ via pull-push $p_* q^*$ or $q_* p^*$

BONDAL ORLOV

The conjecture doesn't provide us with a functor.

Natural candidate: pull-push via the fibre product.

$$\begin{tikzcd}
 & X_- \times_Y X_+ \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X_- \arrow[dr, "f_-"'] & & X_+ \arrow[dl, "f_+"] \\
 & Y &
\end{tikzcd}$$

$\hat{X} := X_- \times_Y X_+$

$\Phi := p_* q^* : D^b(X_+) \longrightarrow D^b(X_-)$

$\Psi := q_* p^* : D^b(X_-) \longrightarrow D^b(X_+)$

Remark: At the moment, I'm ignoring problems that might arise because of the singularities (e.g. Does pullback preserve boundedness?)

When do these functors work?

- ✓ Standard (Atiyah) flops, Mukai flops, CY 3-folds, Abuaf flop, ...
- ✗ Stratified Mukai flops.

Let's say they do work. Then, we have a NON TRIVIAL autoequivalence:

FLOP-FLOP AUTOEQUIVALENCE

$$FF := q_* p^* p_* q^* \in \mathrm{Aut}(D^b(X_+))$$

AUTOEQUIVALENCES OF $D^b(X)$:

Inside $\mathrm{Aut}(D^b(X))$ we always find:

- $[1]$ shift $\left(\cdots \to \underset{\deg i}{E^i} \to E^{i+1} \to \cdots \longmapsto \cdots \to \underset{\deg i-1}{E^i} \to E^{i+1} \to \cdots \right)$
- f_*, where $f \in \mathrm{Aut}(X)$
- $\otimes \mathcal{L}$, where $\mathcal{L} \in \mathrm{Pic}(X)$

They generate a subgroup : $\mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \subseteq \mathrm{Aut}(D^b(X))$

Thm : (Bondal-Orlov) Equality holds if X smooth projective & ω_X is (anti)ample.

CLAIM : When FF is not the identity, we have:

$$FF \notin \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$$

MIRROR SYMMETRY MOTIVATION :

X CY variety $\xleftrightarrow{\text{MIRROR}}$ $\hat{X}$ symplectic variety

Homological mirror symmetry : $D^b(X) \simeq \mathrm{Fuk}(\hat{X})$

This reads the symplectic geometry
(e.g. symplectomorphisms, Dehn twists)

What is FF? Here is where spherical functors come in the game.

Idea : (What's a spherical functor)

REFERENCE : Anno - Logvinenko, Spherical DG-functors

A spherical functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor that satisfies certain properties, and out of which we can construct an autoequivalence $T_F \in \mathrm{Aut}(\mathcal{D})$ called the SPHERICAL TWIST AROUND F.

EXAMPLES:

SPHERICAL TWISTS AROUND SPHERICAL OBJECTS (SEIDEL-THOMAS):

X smooth projective variety, $E \in D^b(X)$ is said to be spherical if

(i) $E \simeq E \otimes \omega_X$ (ii) $\mathrm{Hom}_{D^b(X)}(E, E[i]) = \begin{cases} \mathbb{C} & i = 0, \dim X \\ 0 & \text{otherwise} \end{cases}$

$$\Longrightarrow T_E(F) = \mathrm{cone}\left(R\mathcal{H}om_X(E,F) \otimes E \xrightarrow{ev} F \right)$$

- $T_E(E) \simeq E[1-d]$ $\quad d = \dim X$
- $T_E(F) \simeq F$ $\quad F \in E^{\perp} = \{ G \in D^b(X) : R\mathcal{H}om_X(E,G) = 0 \}$

$$\Rightarrow T_E \notin \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \qquad \begin{matrix} d > 1 \\ E^{\perp} \neq \{0\} \end{matrix}$$

ANY AUTOEQUIVALENCE OF A (NICE) TRIANGULATED CATEGORY:

E. Segal proved in 2016 that any autoequivalence of a (nice) triangulated category can be realized as a spherical twist around a spherical functor.

TENSOR PRODUCT WITH LINE BUNDLE:

$\mathcal{L} \in \mathrm{Pic}(X)$, seek: $F: \mathcal{C} \longrightarrow D^b(X)$ s.t. $T_F = \otimes \mathcal{L}$

$Y = \mathrm{Tot}(\mathcal{L}^\vee) \Rightarrow D^b(Y) \xrightarrow{i_*} D^b(X)$ is spherical and

$$T_{i_*} \simeq \otimes \mathcal{L}[2]$$

(To delete the shift one should consider the sheaf of graded algebras $\mathcal{O}_X \oplus \mathcal{L}^\vee[-1]$).

What's the use? For us the importance of presenting an autoequivalence as the spherical twist around a spherical functor is embodied by the information that $F : \mathcal{C} \longrightarrow \mathcal{D}$ can give us about T_F, e.g.

> Does T_F split as the composition of simpler autoequivalences?

SEMI ORTHOGONAL DECOMPOSITION (SOD):

ANALOGY: SEMIDIRECT PRODUCTS OF GROUPS

Given two groups G, H and a morphism of groups $\phi : G \longrightarrow \mathrm{Aut}(H)$, we can define a new group $H \rtimes_\phi G$ called their semidirect product. This construction exhibits the group $H \rtimes_\phi G$ as built from H and G, but where the choice of which goes first is important.

The idea behind SODs is similar: we want to chop up $D^b(X)$ into smaller pieces which are (semi) orthogonal. Semiorthogonality is defined in terms of:

$$\mathrm{Hom}^\bullet_{D^b(X)}(-,-) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{D^b(X)}(-,-[m])$$

and we say that $D^b(X) = \langle \mathcal{A}_1, -, \mathcal{A}_r \rangle$ is a SOD if:

(i) there are morphisms from $\mathcal{A}_i$ to $\mathcal{A}_j$ iff $i \leq j$

(ii) $D^b(X)$ is generated (in a suitable sense) by $\mathcal{A}_1, -, \mathcal{A}_r$.

EXAMPLES:

PROJECTIVE SPACES (BEILINSON):

We have a SOD: $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$

EXCEPTIONAL COLLECTION:

When we have a sequence of objects $E_1, \dots, E_n$ s.t. $D^b(X) = \langle E_1, \dots, E_n \rangle$ and $\mathrm{Hom}^\bullet_{D^b(X)}(E_i, E_i) = \mathbb{C}$ we call $E_1, \dots, E_n$ a full exceptional collection. Notice that in this case there exist $F_1, \dots, F_n$ s.t. $\mathrm{Hom}^\bullet_{D^b(X)}(F_i, E_j) = \mathbb{C}\,\delta_{ij}$

The F_i's are called the left dual exceptional collection.

BLOW UPS:

$Y \subseteq X$ smooth var., then: $D^b(Bl_Y X) = \langle \underbrace{D^b(Y), D^b(Y), \dots, D^b(Y)}_{\mathrm{codim}_X Y - 1 \text{ copies}}, D^b(X) \rangle$

$0 \in \mathbb{A}^2 \Rightarrow Bl_0 \mathbb{A}^2 = \mathrm{Tot}(\mathcal{O}(-1)_{\mathbb{P}^1}) \xrightarrow{p} \mathbb{A}^2$

$D^b(\mathrm{Tot}(\mathcal{O}(-1)_{\mathbb{P}^1})) = \langle \mathcal{O}_{\mathbb{P}^1}(-1), p^* D^b(\mathbb{A}^2) \rangle$

$\mathrm{Hom}^\bullet_{D^b(Bl_0 \mathbb{A}^2)}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{C} \Rightarrow \langle \mathcal{O}_{\mathbb{P}^1}(-1) \rangle \simeq D^b(pt) \simeq D^b(\mathbb{C})$.

They go together:

Thm: (HALPERN-LEISTNER SHIPMAN) $F: \mathcal{C} \to \mathcal{D}$ a spherical functor, $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ + technical condition. Then:

$$T_F = T_{F|_{\mathcal{A}}} \circ T_{F|_{\mathcal{B}}}$$

Thm: (B.) $F_1: \mathcal{A} \to \mathcal{C}$, $F_2: \mathcal{B} \to \mathcal{C}$ spherical functors. Then, there exists $\tilde{F}: \mathcal{D} \to \mathcal{C}$ spherical functor s.t. (i) $\mathcal{D} = \langle \mathcal{B}, \mathcal{A} \rangle$ (ii) $T_{\tilde{F}} = T_{F_2} \circ T_{F_1}$

Back to our setup:

$$\begin{tikzcd} & \hat{X} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_- \arrow[dr, "f_-"'] & & X_+ \arrow[dl, "f_+"] \\ & Y & \end{tikzcd}$$

ASSUMPTION: $p_* \mathcal{O}_{\hat{X}} \simeq \mathcal{O}_{X_-}$, $q_* \mathcal{O}_{\hat{X}} \simeq \mathcal{O}_{X_+}$ $(*)$

AIM: present $FF = q_* p^* p_* q^*$ as the (inverse) twist around a spherical functor.

Because of the singularities that might appear, we must work with

$D_{qc}(-) = \{$ unbounded complexes with quasi coherent cohomology $\}$

$FF = q_* \, p^* \, p_* \, q^*$

These functors don't see

$K := \operatorname{Ker} p_* \cap \operatorname{Ker} q_* \subset D_{qc}(\hat{X})$

$\Downarrow$

We consider the quotient: $D_{qc}(\hat{X})/_K$.

By assumption $(*)$ the category $D_{qc}(\hat{X})/_K$ has a SOD

$$D_{qc}(\hat{X})/_K = \langle \mathcal{C}, p^* D_{qc}(X_-) \rangle$$

where: $\mathcal{C} = \{ E \in D_{qc}(\hat{X})/_K : p_* E \simeq 0 \}$.

Thm. (B.) The functor $\mathcal{C} \xrightarrow{q_*} D_{qc}(X_+)$ is spherical, and the inverse of the twist around it is $FF \in \operatorname{Aut}(D_{qc}(X_+))$.

REMARKS:

(1): Generally, we are interested in $D^b(X)$ rather than $D_{qc}(X)$. Coming back from the above statement to one about $FF \in \mathrm{Aut}(D^b(X_+))$ is not straightforward, but can be done. In particular, the case when X_+ is smooth can be dealt with quite explicitly.

(2): The idea of studying the quotient category $D_{qc}(\hat{X})/K$ was first pursued by Bodzenta and Bondal in the case where $f_\pm$ have fibres of dimension at most one. They prove:

Thm: (BODZENTA, BONDAL) $D^b(\mathcal{A}_{f_+}) \xrightarrow{i} D^b(X_+)$ is spherical, and $T_i^{-1} \simeq FF$

$\mathcal{A}_{f_+} = \{E \in \mathrm{Coh}(X_+) : (f_+)_* E = 0\}$

The relation to the theorem above is that:

$$\mathcal{C} \simeq D(A_P) \qquad \mathcal{C} \cap D^b(\hat{X})/K^b \simeq D^b(\mathcal{A}_{f_+})$$

endomorphism algebra of projective generator P of $\mathcal{A}_{f_+}$

(3): When $\dim X_\pm = 3$, a result by Donovan-Wemyss says that $FF \simeq T_\Phi^{-1}$, where $\Phi : D^b(A_{con}) \longrightarrow D^b(X_+)$, A_{con} = CONTRACTION ALGEBRA

Bodzenta-Bondal prove that in this case $A_P \simeq A_{con}$

$\Rightarrow \mathcal{C} \simeq D(A_{con})$.

(4): The theorem works for any correspondence $X_- \xleftarrow{p} Z \xrightarrow{q} X_+$ inducing derived equivalences and satisfying $p_* \mathcal{O}_Z \simeq \mathcal{O}_{X_-}$, $q_* \mathcal{O}_Z \simeq \mathcal{O}_{X_+}$.

EXAMPLES:

STANDARD FLOPS:

$X_{\pm} = \mathrm{Tot}(\mathcal{O}(-1)_{\mathbb{P}^n}^{\oplus m+1})$, $\hat{X} = Bl_{\mathbb{P}^n}(X_-) = \mathrm{Tot}(\mathcal{O}(-1,-1)_{\mathbb{P}^n \times \mathbb{P}^n})$

Thm: (ADDINGTON-DONOVAN-MEACHAN)

$\mathcal{O}_{\mathbb{P}^n}$ is a spherical object, and $FF \simeq T^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-1)} \circ T^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-2)} \circ \cdots \circ T^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-m)}$

Given this theorem and the one before, we expect to find a SOD $\mathcal{C} = \langle \underbrace{D(\mathbb{C}), \ldots, D(\mathbb{C})}_{m\text{-copies}} \rangle$, and that $\mathcal{C} \xrightarrow{q_*} D_{qc}(X_+)$ is described by: $\mathbb{C}_i \longmapsto \mathcal{O}_{\mathbb{P}^n}(-i)$ ($\mathbb{C}_i$ is the generator of the i-th copy of $D(\mathbb{C})$ counting right to left).

Thm (B.): In $D_{qc}(\hat{X})/K$ we have morphisms from $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0,-i)$ to $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0,-j)$ iff $i \geq j$ (here $1 \leq i,j \leq m$), and for $i=j$ we have only the identity. Furthermore, the object $\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0,-i)$ is a generator of $\mathcal{C}$. Thus, $\mathcal{C}$ can be thought of as a DG enhancement of the category of modules over the graded algebra (with relations)

$$
\begin{tikzcd}
m \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[rr, bend left] & m-1 \arrow[r, shift left=1] \arrow[r, shift right=1] & m-2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[rr, bend right] & \cdots & 1
\end{tikzcd}
$$

MUKAI FLOPS:

$X_\pm = \mathrm{Tot}(\Omega^1_{\mathbb{P}^n})$, $\quad \hat{X} = \mathbb{P}^n \times \mathbb{P}^n \sqcup_{\mathbb{P}(\Omega^1_{\mathbb{P}^n})} \mathrm{Bl}_{\mathbb{P}^n} X_-$

$$\Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes V^* \longrightarrow \mathcal{O}$$

Thm: (ADDINGTON - DONOVAN - MEACHAN)

$\mathcal{O}_{\mathbb{P}^n}$ is a $\mathbb{P}^n$-object, and $FF \simeq P^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-1)} \circ \dots \circ P^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-n)}$.

(i) $E \simeq E \otimes \omega_X$

(ii) $\mathrm{Hom}^\bullet_{D^b(X)}(E,E) \simeq \mathbb{C}[t]/t^{n+1}$, $\quad |t| = 2$

$\mathbb{P}$-twist around a $\mathbb{P}$-object, introduced by Huybrechts and Thomas.

E. Segal gave a construction of the $\mathbb{P}$-twist around a $\mathbb{P}$-object as the spherical twist around either of the spherical functors:

$$D^b(\mathbb{C}[q]) \xrightarrow[\otimes E]{} D^b(X) \qquad\qquad D^b(\mathbb{C}[\varepsilon]/\varepsilon^2) \xrightarrow[\otimes\, [E[-2] \to E]]{} D^b(X)$$

$\deg q = 2$ $\qquad$ Koszul duality $\qquad$ $\deg \varepsilon = -1$

Similarly to what happened for standard flops, we expect either

$\mathcal{C} = \langle \overbrace{D(\mathbb{C}[q]), \_, D(\mathbb{C}[q])}^{n\text{-copies}} \rangle$, or $\mathcal{C} = \langle \overbrace{D(\mathbb{C}[\varepsilon]/\varepsilon^2), \_, D(\mathbb{C}[\varepsilon]/\varepsilon^2)}^{n\text{-copies}} \rangle$.

Thm (B.): We have a SOD $\mathcal{C} = \langle \overbrace{D(\mathbb{C}[\varepsilon]/\varepsilon^2), \_, D(\mathbb{C}[\varepsilon]/\varepsilon^2)}^{n\text{-copies}} \rangle$ where the generator is given by the projection to $\mathcal{C}$ of the pull up to X_+ and then to $\hat{X}$ of $\bigoplus_{j=1}^{n} \Lambda^j(\Omega^1_{\mathbb{P}^n}(-1))[-j]$. Moreover there is a subcategory $\mathcal{T} \subseteq \mathcal{C}$ s.t. $\mathcal{T} = \langle \overbrace{\mathrm{Perf}(\mathbb{C}[q]), \_, \mathrm{Perf}(\mathbb{C}[q])}^{n\text{-copies}} \rangle$ and whose generator is given by $\bigoplus_{j=1}^{n} \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0,-j)$.

$\mathcal{T}$ can be thought of as a DG enhancement of category of modules over

$$\begin{tikzcd} \underset{\mathbb{C}[q]}{n} \arrow[loop above] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[rr, bend left] & \underset{\mathbb{C}[q]}{n-1} \arrow[loop above] \arrow[r, bend right] & \cdots \quad \underset{\mathbb{C}[q]}{1} \arrow[loop above] \end{tikzcd}$$

$\mathcal{C}$ can be thought of as a DG enhancement of category of modules over

$$\begin{tikzcd} \underset{\mathbb{C}[\varepsilon]/\varepsilon^2}{n} \arrow[loop above] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[rr, bend left] & \underset{\mathbb{C}[\varepsilon]/\varepsilon^2}{n-1} \arrow[loop above] \arrow[r, bend right] & \cdots \quad \underset{\mathbb{C}[\varepsilon]/\varepsilon^2}{1} \arrow[loop above] \end{tikzcd}$$

OTHER EXAMPLES:

The therem has only a few hypotheses, hence it can be applied in many cases, e.g. Grassmannian flops, Abuaf flop to name some.

Each of these two examples present interesting features:

- Grassmannian flops are a generalization of standard flops, but the fact that the GIT problem in this case has more than one stratum changes everything.
- The Abuaf flop is an example in which the flop-flop autoequivalence is the composition of inverses of spherical twists around spherical objects which are not independent in the K-theory group.

Thank you!