Successive minima of line bundles

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(based on joint work with Atsushi Ito, Adv. Math 305, 2020)

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Aim of talk:

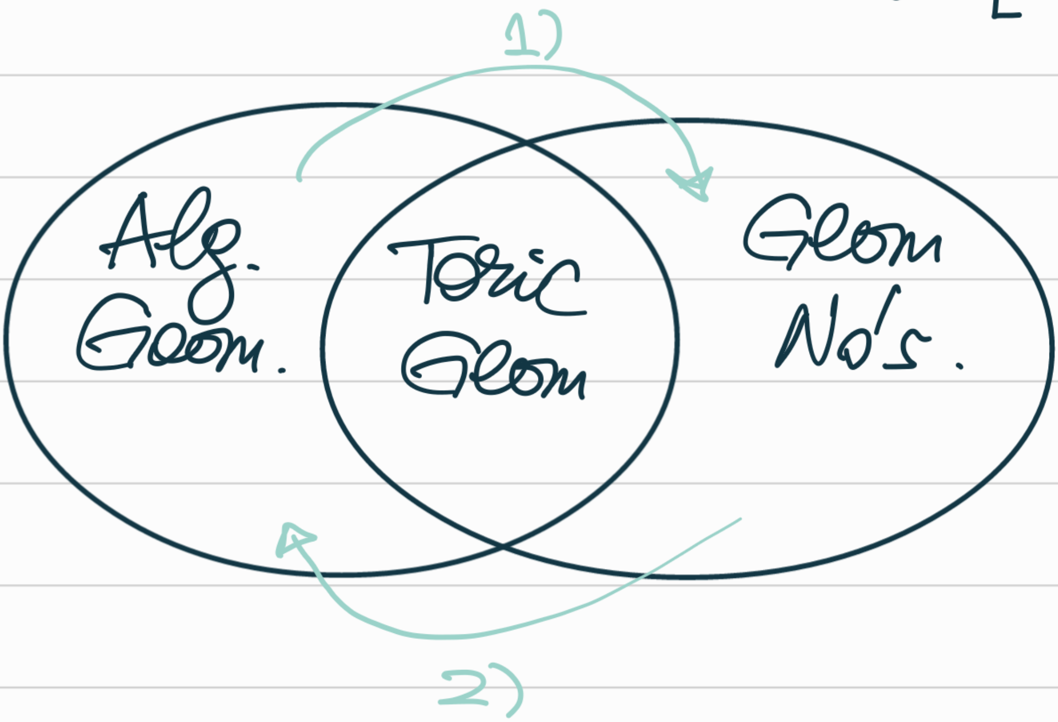
1) Control Seshadri const. of l.b. on toric var's.

$\Rightarrow$ gen'n of Khinchine's Flatness Thm.

2) Extend Sesh. ct to a sequence of invariants

$$\underset{\varepsilon(L)}{\underset{\shortparallel}{\varepsilon_d}} \le \varepsilon_{d-1} \le \dots \le \underset{w(L)}{\underset{\shortparallel}{\varepsilon_1}}$$

- $\mathrm{vol}(L) \approx \varepsilon_1 \cdots \varepsilon_d$ (AG analogue of Mink Thm II)
- (X, L) toric $\Rightarrow \varepsilon_i(X, L) \approx 1/\lambda_i(\square_L - \square_L, M)$.



Outline of talk:

1. GN (Minkowski; Transf. FT)
2. Adjoint linear systems, Seshadri cts.
3. Sesh. cts on toric var's
4. Successive minima of l.b's.
5. Questions, problems

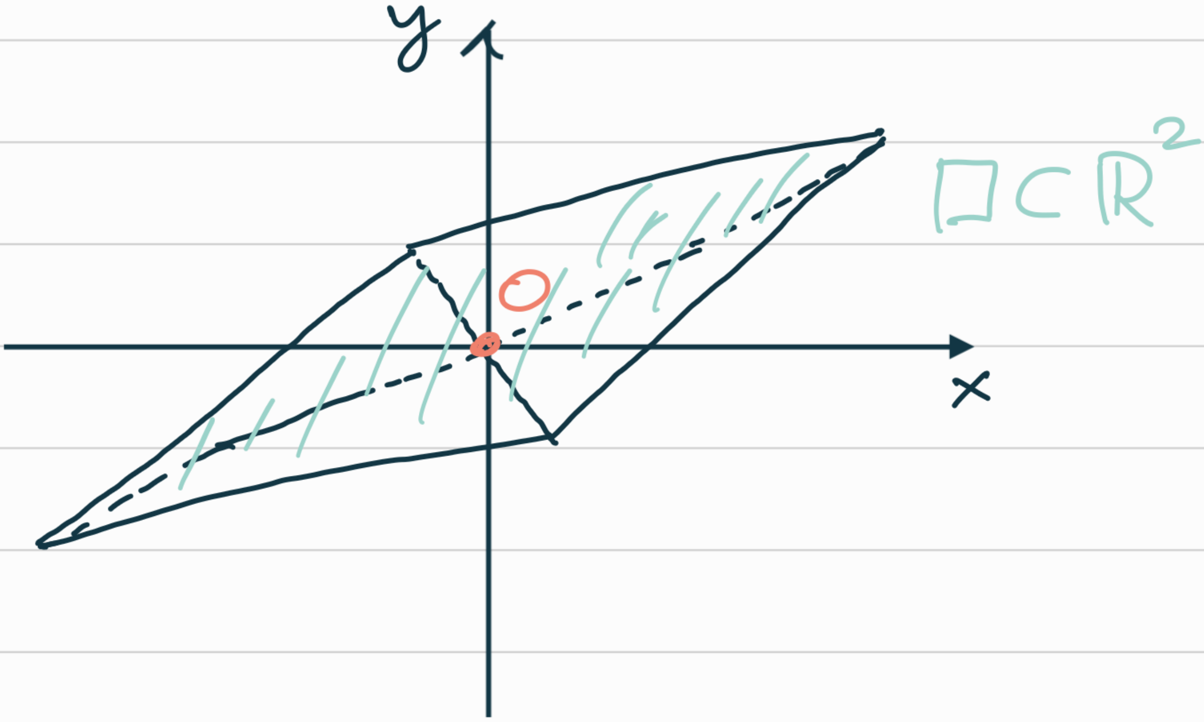
① Geometry of numbers

$\Lambda \simeq \mathbb{Z}^d$ lattice of dim. d.

$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \supset \square$ convex, compact, dim d.

a) Case $\square$ is 0-symmetric:

$x \in \square \Leftrightarrow -x \in \square \quad \Rightarrow 0 \in \mathrm{Int}\,\square = \mathring{\square}$



Thm I (Minkowski) $\Lambda \cap \mathring{\square} = \{0\} \Rightarrow \mathrm{vol}_\Lambda \square \leq 2^d$.

$\lambda_1(\Lambda, \square) = \inf\{t > 0 ; \Lambda \cap t\square \neq \{0\}\}$.

$\lambda_i(\Lambda, \square) = \inf\{t > 0 ; \dim(\Lambda \cap t\square) \geq i\}$.

$0 < \lambda_1 \leq \ldots \leq \lambda_d$ successive minima $(\Lambda, \square)$.

Th I $\iff \lambda_1^d \cdot \mathrm{vol}_\Lambda \square \le 2^d$.

Th II (Minkowski) $\dfrac{2^d}{d!} \le \lambda_1 \cdots \lambda_d \cdot \mathrm{vol}_\Lambda \square \le 2^d$.

$\Lambda^* := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ dual lattice

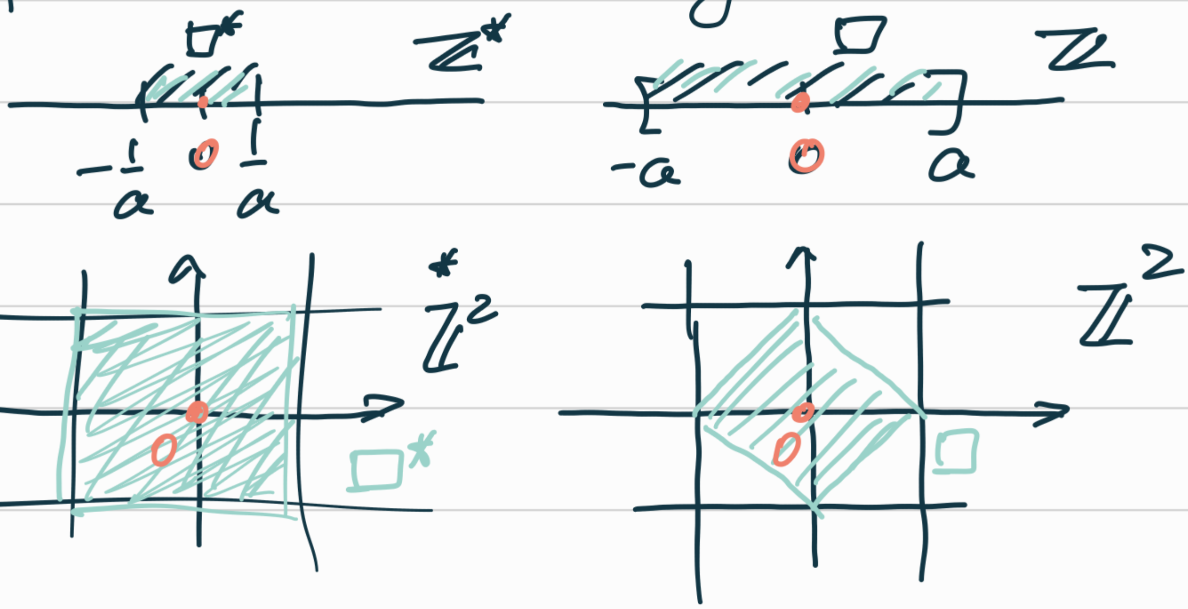
$\Lambda^* \times \Lambda \to \mathbb{Z}$ duality pairing.

$(\varphi, \lambda) \mapsto \varphi(\lambda)$

$\square^* = \{\varphi \in \Lambda^*_{\mathbb{R}} \,;\, \langle \varphi, \lambda \rangle \ge -1 \ \forall \lambda \in \square\}$

polar (dual) body.

Exa:



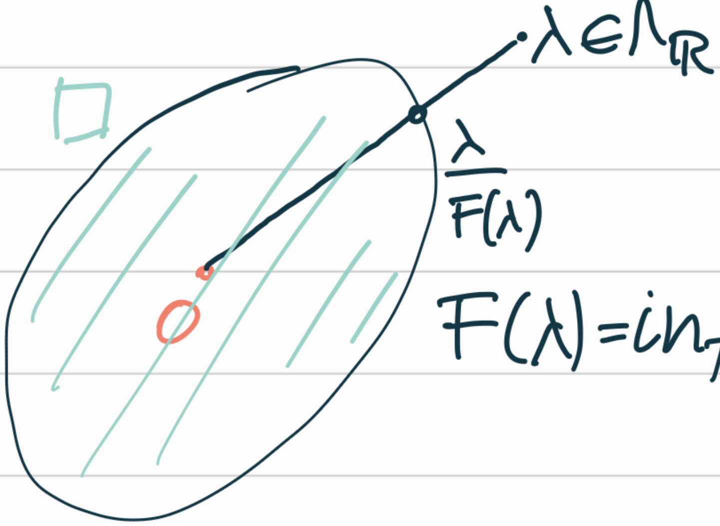
small vs. big.

Th Transference (Mahler, Banaszczyck):

$1 \le \lambda_i(\Lambda^*, \square^*) \cdot \lambda_{d-i+1}(\Lambda, \square) \le d \quad \forall\, i = 1, .., d$.

Applic'ns: number theory, dioph. approx'n.

- How to compute succ. min?



$F(\lambda) = \inf\{t \geq 0;\ \lambda \in t\square\}$.

$$F : \Lambda_{\mathbb{R}} \to [0,\infty) : \quad F(x) = F(-x)$$
$$F(cx) = cF(x) \quad \forall c > 0$$
$$F(x+y) \leq F(x) + F(y).$$

$$\square = \{x \in \Lambda_{\mathbb{R}} ;\ F(x) \leq 1\}$$

$$\lambda_1(\Lambda, \square) = \min_{0 \neq \lambda \in \Lambda} F(\lambda)$$

- Davenport–Estermann extended Th II Mink. to <u>ANY</u> cv. body $\square \subset \Lambda_{\mathbb{R}}$:

$$\frac{1}{d!} \leq \prod_{i=1}^{d} \lambda_i(\Lambda, \underbrace{\square - \square}_{\text{0-symm.}}) \cdot \mathrm{vol}_{\Lambda} \square \leq 1.$$

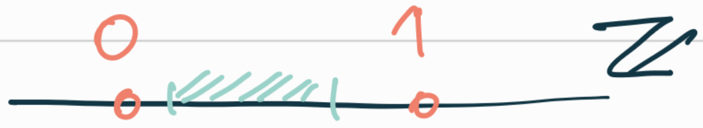
0-symm.

($= 2\square$ if $\square \ni 0$ symm.)

b) Case $\square \subset \Lambda_{\mathbb{R}}$ arbitrary conv. body:

Q: $\Lambda \cap \mathring{\square} = \emptyset$ can classify?

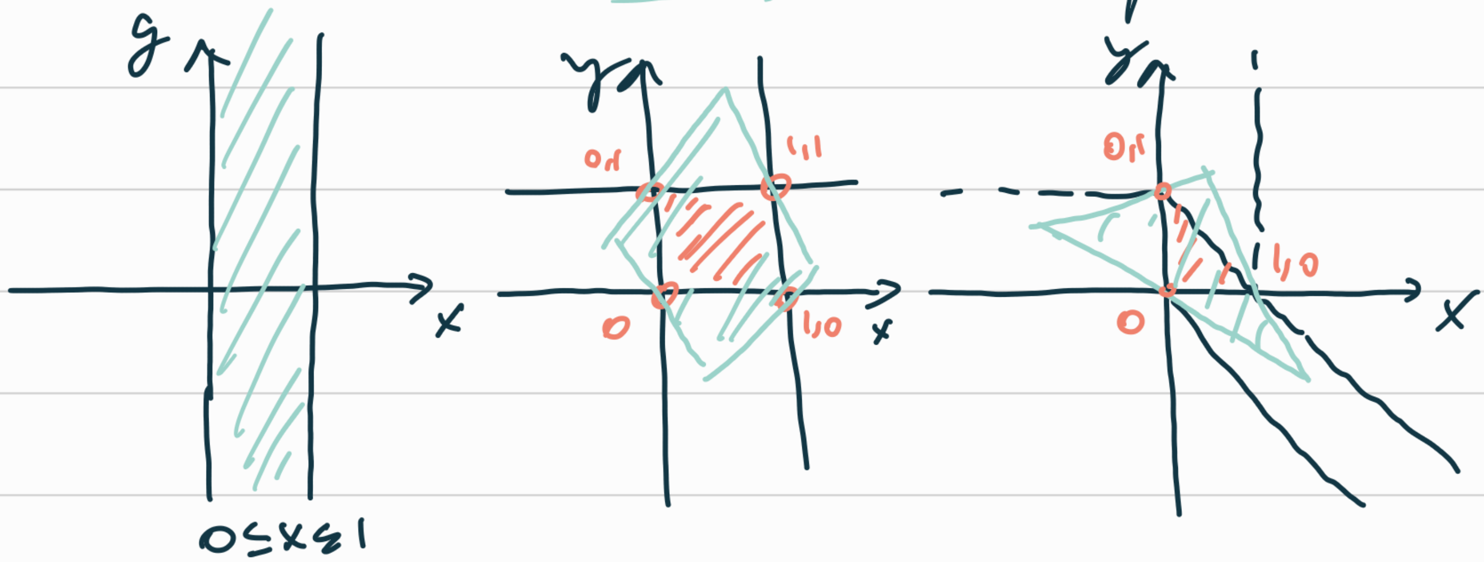
$d=1$:



$\square + z$ ($\exists z \in \mathbb{Z}$).

$\Rightarrow$ length $\square \le 1$.

$d=2$: $\lambda + \square \subset \underline{\square}^{max}$ of 3 types:



$vol_\Lambda \square \to +\infty$ possible!

But $\exists$ lattice proj'n of length $\le 1 + \frac{2}{\sqrt{3}}$.
(Hurkens)

$d \ge 3$?

Flatness Thm (Khinchine) $\Lambda^d \cap \mathring{\square} = \emptyset \implies$
$\exists\, \varphi : \Lambda \twoheadrightarrow \mathbb{Z}$, length $\varphi_{\mathbb{R}}(\square) \leq C_d$.

$(\square - \square)^* = \{ \varphi \in \Lambda^*_{\mathbb{R}} ; \text{length } \varphi(\square) \leq 1 \}$

$\lambda_1(\Lambda^*, (\square - \square)^*) = \min_{0 \neq \varphi \in \Lambda^*} \text{length } \varphi_{\mathbb{R}}(\square) =:$ width$(\Lambda, \square)$
lattice width of $\square$.

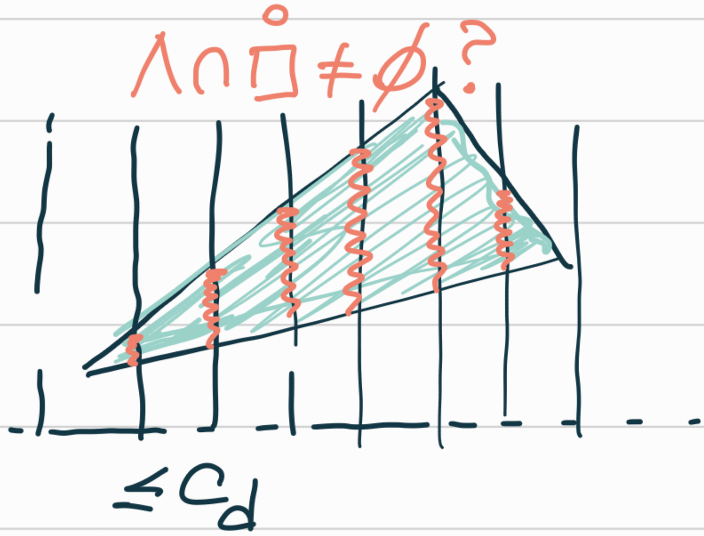
FT: $\Lambda \cap \mathring{\square} = \emptyset \implies \text{width}(\Lambda, \square) \leq C_d$.

C_d:
- $\approx d!$ Khinchine 1948
- c^{d^2} Lenstra 1983
- $c \cdot d^2$ Kannan + Lovasz 1988
- $c \cdot d^{3/2}$ Banaszczyck 1999
- Q: linear in d?

- K-L: $\sqrt[d]{|\Lambda \cap \mathring{\square}|} \geqslant \dfrac{w(\Lambda, \square)}{c \cdot d^2} - 1$.

Applications:

- integer linear programming method:



- Sol'ns in toric case of some open problems in ber'l classif. of alg var's.

Q: $\exists$ AG analogue of PT? Yes, even stronger!

- AG interpret'n of interior lattice pts:

$$\begin{array}{ccc} (\Lambda^d, \square) & \mathbb{Q}\text{-polytope} & \\ \downarrow & & \text{pullback of } L \\ (X^d; L) & \longleftarrow & (X'; L') \\ \text{toric proj} \quad \mathbb{Q}\text{-polar.} & & \text{toric resolution} \end{array}$$

$$\Gamma(X', \lceil K_{X'} + L' \rceil) = \bigoplus_{\lambda \in \Lambda \cap \mathring{\square}} \mathbb{C} \cdot \chi^\lambda$$

② Adjoint linear systems & Seshadri constants

$(X^d; L)$ proj smooth$/\mathbb{C}$ + ample l.b.

Q: $|K_X+L| \neq \emptyset$?

$\phi_{|K_X+L|}: X \dashrightarrow X' \subset \mathbb{P}$ $\dim X' \geq i$? ϕ bir'l?

Heuristic: NO $\Rightarrow$ $(X, L) \ni$ special cycles.

Fujita Conj: $|K_X+nL|$ base point free $n \geq d+1$

very ample $n \geq d+2$.

Demailly '92: OK if at any given pt $x \in X$, L admits a positive hermitian metric with: a) isolated singularity at x; b) large Lelong no. at x.

Example thm: $x_1, \dots, x_a \in X$ distinct

$p_1, \dots, p_a, q \in \mathbb{Z}_{\geq 1}$.

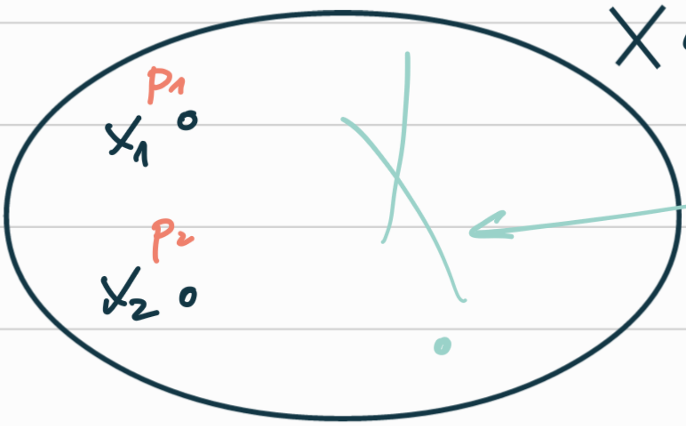
$\dim_{x_i} \mathrm{Bs}\,|I_{x_1}^{p_1} \otimes \cdots \otimes I_{x_a}^{p_a}(qL)| = 0 \ \forall i = 1, a$

$\Rightarrow \Gamma(K_X+L) \twoheadrightarrow \bigoplus_{i=1}^{a} J_{x_i}^{\lceil \frac{p_i}{q} - d - 1 \rceil}$

$J_x^l := \mathcal{O}_{X,x}/\mathfrak{m}_x^{l+1}$ jets of order l at x.

Proof: Basis of $\Gamma(I_{x_1}^{p_1}\otimes\cdots\otimes I_{x_a}^{p_a}(qL)) \rightsquigarrow h$ on L
pos. sing herm metric, smooth on punctured nbhd of each x_i, with Lelong no $\geq \frac{p_i}{q}$ at x_i.
Apply Nadel vanishing $\boxtimes$

$$X \overset{|qL|}{\hookrightarrow} \mathbb{P}$$



unassigned base locus should be away from x_i!

DEF (Demailly) Seshadri const of (X,L) at x:

- $\Gamma(L^n) \twoheadrightarrow J_x^{j_n}$ ← max'l (n) $\quad \lim_{n\to\infty} \frac{j_n}{n} =: \varepsilon(L,x)$.

- $\begin{matrix} Bl_x X & \supset & E \\ \downarrow f & & \downarrow \\ X & \ni & x \end{matrix}$ $\quad \varepsilon(L;x) = \sup\{t>0;\ f^*L - tE \text{ ample}\}$
$$= \inf_{C \ni x} \frac{L\cdot C}{\mathrm{mult}_x C}.$$

- $\varepsilon(L,x) = \sup\left\{\frac{p}{q};\ \exists D_1,\dots,D_d \in |L^q|,\ \mathrm{mult}_x D_i \geq p,\ \underbrace{D_1\cap\cdots\cap D_d = \{x\} \text{ near } x}_{\Leftrightarrow\ Bs|I_x^p \otimes L^q| = x \text{ near } x.}\right\}$

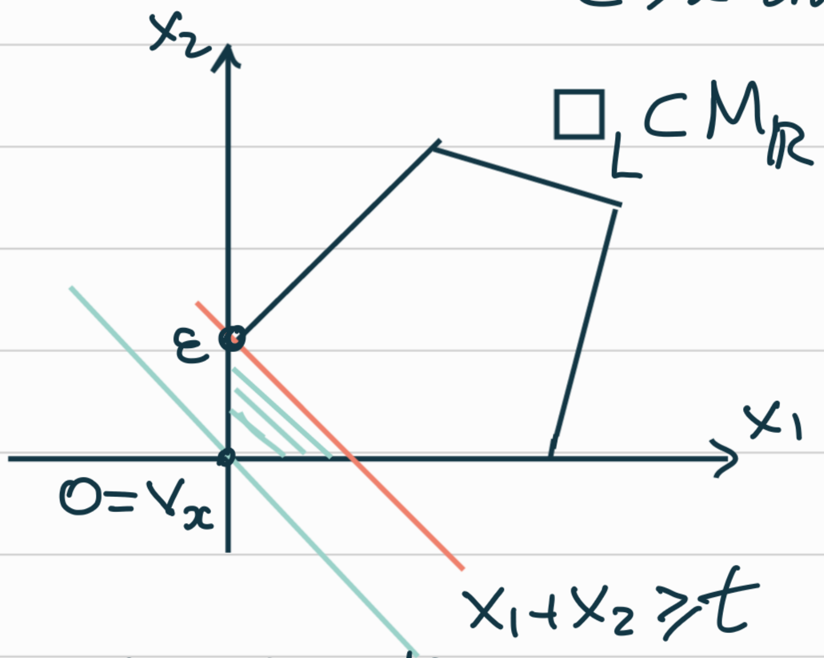
$$\Gamma(K_X + L) \twoheadrightarrow J_x^{\lceil \varepsilon(L,x) \rceil - d - 1}$$
Need lower bds!!

- $X \ni x \mapsto \varepsilon(L;x)$ attains max'l value for x VERY general. "$\varepsilon(L)$"
- $\varepsilon(L^n,x) = n\,\varepsilon(L,x)$, $\varepsilon(L_1+L_2,x) \geq \varepsilon(L_1,x)+\varepsilon(L_2,x)$.
- $\mathbb{P}^d$, $\mathcal{O}(1) \rightsquigarrow \varepsilon(x)=1 \ \forall x$.

  $\dim X = 1$, $L>0 \rightsquigarrow \varepsilon(x) = \deg L \ \forall x$

  $\varepsilon(L,x)$ hard to compute in gen'l!
- (Rocco, '99) (X,L) toric polar, x inv. pt

  $\Rightarrow \varepsilon(L,x) = \min\limits_{C \ni x \text{ inv. cv.}} (L\cdot C)$



$\Gamma(L^n) = \bigoplus\limits_{m \in M \cap n\square_L} \mathbb{C}\cdot X^m$

- $\varepsilon(L,x) \leq \sqrt[d]{\mathrm{vol}(L)}$.
- Q (cf. Nakamaye, Hwang-Keum)

  $\varepsilon(L)/\sqrt[d]{v(L)} \approx 0 \Rightarrow \exists X \dashrightarrow Y$ nontrivial fibr'n, $\varepsilon(L) = \varepsilon(L|_F)$?

- X^2: $\varepsilon(L) \geq 1$ (E-L '93). But $\varepsilon(L,x) \downarrow 0$ possible (Miranda).
- (EKL) $\varepsilon(L) \geq \frac{1}{d} \min_{Y \subseteq X \text{ cycle}} \mathrm{vol}(L|_Y)^{1/\dim Y}$.

should be 1?

Th (Demailly '92; Ein-Küchle-Lazarsfeld '95)
$X^d/\mathbb{C}$ smooth proj, L nef big $\mathbb{Q}$-Ca div, Supp$\{L\}$ N.C.

$x \in X$ very gen'l $\to \varepsilon(L,x) = \varepsilon(L) =: \varepsilon$.

1) $\Gamma(\ulcorner K_X + L \urcorner) \twoheadrightarrow \mathcal{J}_x^{\ulcorner \varepsilon - d - 1 \urcorner}$.

$$\Rightarrow \dim_{\mathbb{C}} \Gamma(\ulcorner K_X + L \urcorner) \geq \binom{\ulcorner \varepsilon - 1 \urcorner}{d}$$

2) $\varepsilon > d \Rightarrow |\ulcorner K_X + L \urcorner| \neq \emptyset$

3) $\varepsilon > d+1 \Rightarrow \dim \Phi_{|\ulcorner K_X + L \urcorner|}(X) = d$.

4) $\varepsilon > 2d \Rightarrow X \dashrightarrow \Phi_{\ulcorner K_X + L \urcorner}(X)$ bir'l.

Proof: Kawamata-Viehweg vanish on blow-up of X $\boxtimes$

Miranda's exa $\Rightarrow$ Demailly's approach to Fuj. Conj using Sesh cts seems to work only at VERY gen'l pts.

③ Seshadri constants on toric var's.

(X^d, L) toric polarized var.

$X \ni x \mapsto \varepsilon(L, x)$ const. on torus orbits

x invar pt $\Rightarrow \varepsilon(L, x)$ known (Rocco)

$\exists$ formula $\forall x$, in terms of invariant and gen'l pts (Ito)

Thm 1 $\square_L \subset M_{\mathbb{R}}$ moment polytope, $w := \mathrm{width}(M, \square_L)$.

$$\frac{w}{d} \le \varepsilon(L) \le w.$$

COR: $\Lambda \simeq \mathbb{Z}^d$, $\square \subset \Lambda_{\mathbb{R}}$ cv body width $= w$:

- $w > d^2 \Rightarrow \Lambda \cap \mathring{\square} \ne \emptyset$ $c = 1$ for K-L.
- $w > d^2 + d \Rightarrow \dim \Lambda \cap \mathring{\square} = d$ gen'n of FT
- $w > 2d^2 \Rightarrow \Lambda \cap \mathring{\square}$ spans Λ cf. AHN
- $\sqrt[d]{d! \, |\Lambda \cap \mathring{\square}|} \ge w/d - d$.
- $\sqrt[d]{d! \, \mathrm{vol}_\Lambda \square} \ge w/d$.

$\because$/ By thm 1 + Dem-EKL thm $\boxtimes$

Proof of Thm 1:

"$\varepsilon(L) \leq w$": $0 \neq \varphi \in M^* = N$.

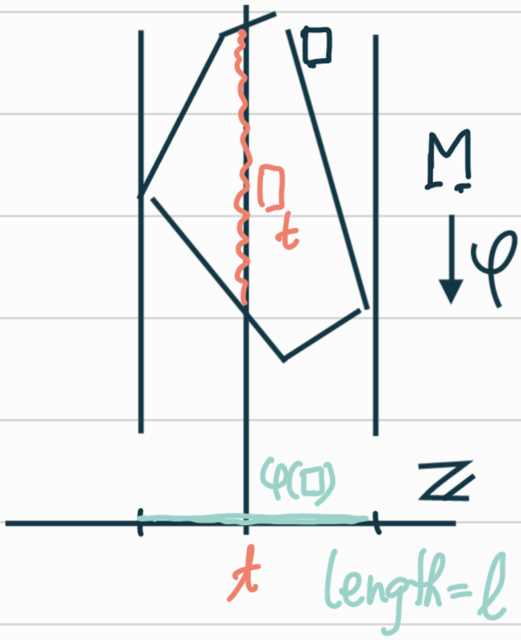
$\square_t \subset \square \rightsquigarrow (X, L) \dashrightarrow (X_t^{d-1}, L_t)$

fibr'n

$(F, L|_F) \simeq (\mathbb{P}^1, \mathcal{O}(\ell))$

$\varepsilon(X, L) \leq \varepsilon(F, L|_F) = \ell$

$\min(\ell) = w(M, \square_L)$.



Transf. thm.

"$\frac{w}{d} \leq \varepsilon(L)$": $w = \lambda_1((\square_L - \square_L)^*) \leq \dfrac{d}{\lambda_d(\square_L - \square_L)}$

Enough $\underline{\varepsilon \geq 1/\lambda_d(\square_L - \square_L)}$:

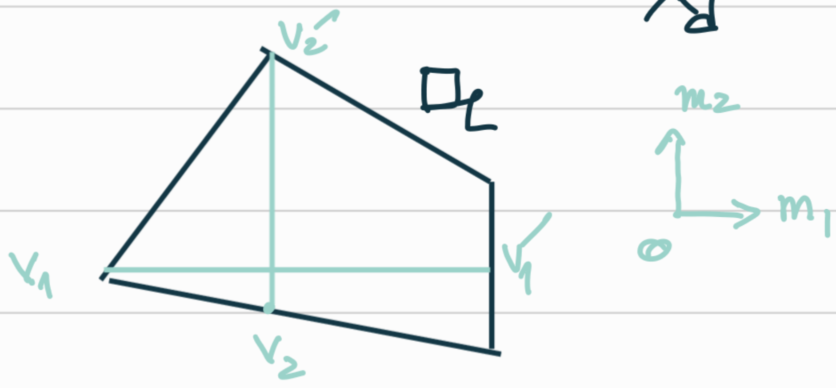
$\exists\, m_1, \dots, m_d \in M^{prim}, \ell_i,\ m_i = \lambda_d(v_i' - v_i),\ v_i, v_i' \in \square$.

$[v_i, v_i'] \subset \square_L \rightsquigarrow (X, \lambda_d L) \overset{dom}{\underset{f_i}{\dashrightarrow}} (\mathbb{P}^1, \mathcal{O}(1))$

$\Rightarrow \exists\, F_i + D_i \in |\lambda_d L|_{\mathbb{Q}}$

$F_i = T_N \cap m_i^{\perp} \quad D_i \not\ni 1$.

$F_1 \cap \dots \cap F_d = \{1\}$ near 1. $\Rightarrow \frac{1}{\lambda_d} \leq \varepsilon$ $\boxtimes$



4 Successive minima of line bundles.

- (X^d, L) toric $\to \varepsilon(L) \approx 1/\lambda_d(\square_L - \square_L, M)$.
  proof involves λ_1^*, λ_d.
  Q: All $\lambda_i(\square_L - \square_L, M)$ have AG meaning? Yes!

- X^d/k proper alg var (irr + reduced), L l.b. on X.
  $x \in X$ $t \geq 0 \rightsquigarrow \mathrm{Bs}|\mathcal{I}_x^{t+} L|_{\mathbb{Q}} := \bigcap \left\{ Z(s) ;\ s \in \Gamma(L^n), n \geq 1, \ \mathrm{ord}_x(s) > nt \right\}$
  closed $\subset X$, $\uparrow$ wrt t, $= X$ for $t \gg 0$.
  $i \geq 1 \rightsquigarrow \varepsilon_i(L, x) := \inf\{t \geq 0 ; \mathrm{codim}_x \mathrm{Bs}|\mathcal{I}_x^{t+} L|_{\mathbb{Q}} < i\}$
  the i-th succ. min of (X, L) at x.
  $\varepsilon_1(L, x) \geq \varepsilon_2(L, x) \geq \cdots \geq \varepsilon_d(L, x) \geq 0 = \varepsilon_{d+1}(L, x)$

- $x \mapsto \varepsilon_i(L; x)$ const. for x very gen'l $\rightsquigarrow \varepsilon_i(L)$.
  $\kappa(L) \leq 0 \implies \varepsilon_1(L) = 0$
  $\kappa(L) = \kappa \geq 1 \implies \varepsilon_\kappa(L) > 0 = \varepsilon_{\kappa+1}(L)$.
  L big $\implies \varepsilon_i(L)$ depends only on numer. class(L).

- L semia, x smooth $\Rightarrow \varepsilon_d(L, x) =$ Sesh. const. $\varepsilon(L, x)$ of Demailly.

L semia $\Rightarrow \varepsilon_d(L) = \max$'l Sesh. ct. $\varepsilon(L)$.

• $\varepsilon_1(L,x) = \sup\left\{ \dfrac{\mathrm{ord}_x(s)}{n} \,;\, n \geq 1, s \in \Gamma(L^n) \right\}$

width of L at x, $w_x(L)$.

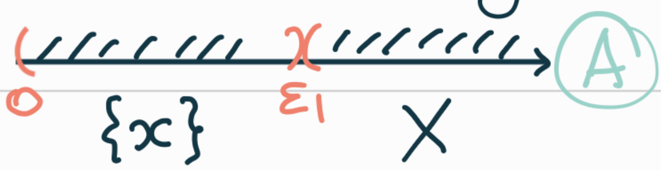
$$\varepsilon_d(L,x) \leq \sqrt[d]{\frac{\mathrm{vol}(L)}{\mathrm{mult}_x X}} \leq \varepsilon_1(L,x)$$

analogue of Mink. Th I.

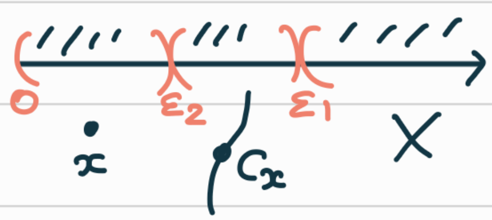
Cor: $\dim X = 1$, $L > 0 \Rightarrow \varepsilon_1(L,x) = \dfrac{\deg L}{\mathrm{mult}_x X}$.

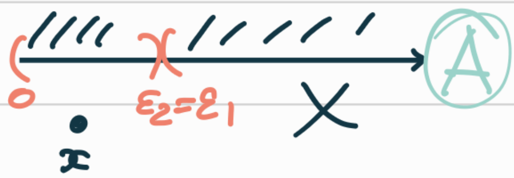
• $t \mapsto \mathrm{Bs}\| I_x^{t} L \|_{\mathbb{Q}}$ say L ample near x.

$d=1$

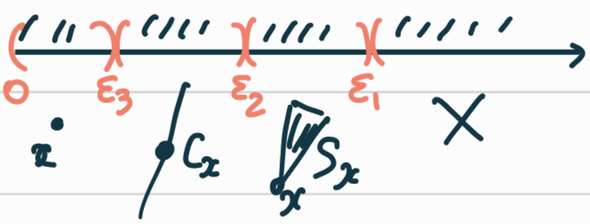


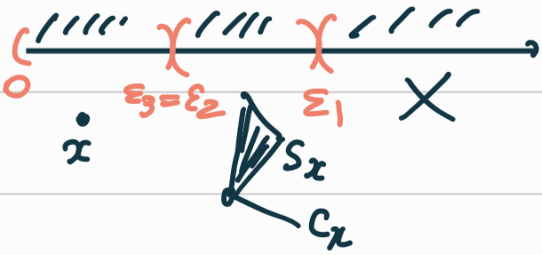
$d=2$

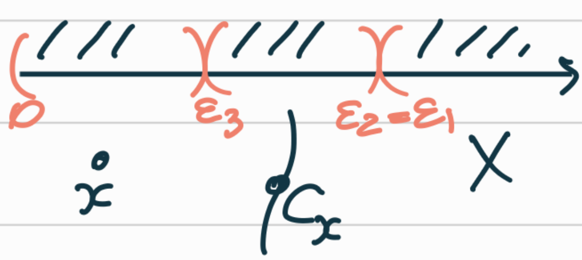


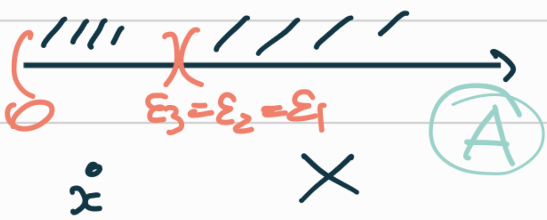


$d=3$









Dichotomy

- $\varepsilon_d = \cdots = \varepsilon_1$ "atomic" case
- $\exists\, \varepsilon_i \neq \varepsilon_j$: x v. gen $\Rightarrow X$ admits foliation (fibr'n?) leaves bnd? quot atomic?

Prop (EKL, Nakamaye) char$(k)=0$!!!

$x \in X$ $\boxed{\text{VERY}}$ gen'l, $t \geq 0$

$Z \subset \mathrm{Bs}\| I_x^{t+} L \|_{\mathbb{Q}}$ irred compon $\ni x$.

$s \in \Gamma(I_x^p(qL))$, $p, q \in \mathbb{Z}_{\geq 1} \Rightarrow \mathrm{ord}_Z(s) \geq p - qt$.

Thm 2 X^d/k char$=0$! proper alg var, L lb $\Rightarrow$

$$\prod_1^d \varepsilon_i(L) \leq \mathrm{vol}(L) \leq d! \cdot \prod_1^d \varepsilon_i(L).$$

analogue of Mink. th II

COR: L big $\Rightarrow$ $1 \leq \dfrac{\mathrm{vol}\, L}{\varepsilon_d(L)^d} \leq d! \left(\dfrac{\varepsilon_1(L)}{\varepsilon_d(L)} \right)^d$.

Thm 3 $(X^d = T_N \operatorname{emb}(\Delta), L)$ toric proper

$\varepsilon_i = \varepsilon_i(L)$, $\lambda_i = \lambda_i(\square_L - \square_L, M)$, $\lambda_i^* = \lambda_i((\square_L - \square_L)^*, N)$.

$$1 \leq \varepsilon_i \lambda_i \leq d \frac{\varepsilon_i}{\lambda^*_{d-i+1}} \leq d(d-i+1) \quad \forall i = 1, .., d.$$

So $\varepsilon_i \approx \dfrac{1}{\lambda_i} \approx \lambda^*_{d-i+1}$

About proofs:

- th 3 similar to th 1.
- th 2 : LHS easy (classical jet counting) holds $\forall x$!

RHS : fails if x special

$\Uparrow$

$$\dim \Gamma(L) \leq |\mathbb{Z}^d \cap \square(\varepsilon_1, .., \varepsilon_d)|$$

$$\square(t_1, .., t_q) = \bigcap_{i=1}^{q} \{x \in \mathbb{R}^d_{\geq 0} ; x_i + \dots + x_d \leq t_i\}$$

$$\mathrm{vol}\, \square(t_1, .., t_q) \leq t_1 \cdots t_q$$

$\Uparrow$

$$h^0(I_x^p L) - h^0(I_x^{p+1} L) \leq \left|\bigcap_{i=2}^{d} \{\alpha \in \mathbb{N}^d(p) ; \alpha_i + \dots + \alpha_d \leq \varepsilon_i\}\right|$$

$$0 \to \Gamma(I_x^{p+1}(L)) \to \Gamma(I_x^p(L)) \xrightarrow{r} I_x^p / I_x^{p+1} \simeq \Gamma \mathcal{O}_{\mathbb{P}^{d-1}}(p).$$

$\leftarrow$ cod in X.

$1 < i \leq d$, $\varepsilon_i < p, \varepsilon_{i-1}$: $\exists\, x \in Z^{i-1} \subset Bs|I_x^{\varepsilon_i +} L|_{\mathbb{Q}}$ irr. comp

$Bl_x X \to X$ $\tilde{Z} \cap E_x \supset W^{i-1} \subset \mathbb{P}^{d-1}$ irr comp, cod $= i-1$

$s \in \Gamma(I_x^p L)$: $E \& L - N \Rightarrow \mathrm{ord}_Z s \geq p - \varepsilon_i$

$\Rightarrow \mathrm{Im}(r) \subset \{P ; \mathrm{ord}_{W^{i-1}}(P) \geq p - \varepsilon_i\}$.

Reduced to :

Prop $\mathbb{P}^d \supset Z^1, Z^d$ irr. subv, $\operatorname{cod} Z^i = i$

$p_1, \dots p_d, q \in \mathbb{Z}_{\geq 0}$.

$\Rightarrow \dim \Gamma(\mathbb{P}^d, \bigcap_{i=1}^{d} I_{Z^i}^{(p_i)}(q)) \leq |\mathbb{Z}^d \cap \square(q-p_1, \dots, q-p_d)|$

Equality holds if $Z^1 \supset Z^2 \supset \dots \supset Z^d$ linear flag

$\because$ Cut with gen'l hyp section, use ind on dim $\boxtimes$

$I_{Z \subset \mathbb{P}}^{(p)} = \{ f \in \mathcal{O}_{\mathbb{P}} ; \operatorname{ord}_x f \geq p \ \forall x \in Z \}$

p-th symbolic power of $I_{Z \subset \mathbb{P}}$.

COR (implicit in Nakayama)

$\dim X = 2$, L ample $\Rightarrow (L^2) \geq 2\varepsilon_1 \varepsilon_2 - \varepsilon_2^2$.

$\because$

$2 \operatorname{vol} \square(t_1, t_2) = 2t_1 t_2 - t_2^2$

$\boxtimes$

5) Questions/problems:

- (X,L) toric: $\varepsilon(L)$ computable? $\in \mathbb{Q}$?
  $\varepsilon_i(L,x)\ \forall x$: estimate/compute?

- $\{(X^d, L);\ w(L) \leq t\}$ "bounded"?

- (X^d, L): a) L nef, $\kappa(L) = \kappa \geq 1 \overset{?}{\Rightarrow} \varepsilon_\kappa(L) \geq 1$ cf. EKL.
  b) $w(L|_Y) \geq 1\ \forall Y \subseteq X \overset{?}{\Rightarrow} \varepsilon(L) \geq 1$
- (X^d, K_X) lc model: $\varepsilon_1(K) \geq \frac{1}{u_{d+2}}$, $\varepsilon_d(K) \geq \frac{1}{u_{d+3}}$?
  $u_1 = 1,\ u_{n+1} = u_n(1+u_n)$

- If >0, $\varepsilon_i(L,x) = \sup\left\{\frac{p}{q} ;\ \operatorname{cod}_x \operatorname{Bs}|I_x^p(qL)| \geq i\right\}$
  When sup = max? Yes $\Rightarrow \in \mathbb{Q}$
  True if X Fano / L adjoint?

- AG analogue of Mink Th II: what if $\operatorname{char}(k) > 0$?

- AG analogue of Transf. Thm?

- Study dichotomy $\varepsilon_d = \varepsilon_1$ | $\varepsilon_d < \varepsilon_1$

special? foliation; relate to Seshadri-exc foliations of Hwang-Keum

- $\dfrac{\varepsilon_1(L)}{\varepsilon_d(L)} \geq C_d \overset{?}{\Longrightarrow} \exists\, X \dashrightarrow Y$ rat'l fibr'n

  F = norm'n of gen'l fiber

  $0 < \dim F < \dim X$

  $\varepsilon(X;L) = \varepsilon(F, L|_F)$

cf. Nakamaye, Hwang-Keum.

GN analogue?

- $\mathrm{Bs}\,|\mathcal{I}_x^{\varepsilon(L,x)+} L|_{\mathbb{Q}} = \bigcup$ "min'l" cycles $\ni x$;

  have bnd L-deg?