Hodge numbers are not derived invariants in char p.

Nick Addington, joint w/ Dan Bragg arXiv: 2106.09949

Why this question?

Derived cats of coh sheaves on sm. proj. vars / field k

(today: $k = \mathbb{C}$ or $\overline{\mathbb{F}}_p$)

$D^b(X)$ behaves like $H^\bullet(X)$

① if $p : P \to X$ is a $\mathbb{P}^n$ bundle

then $H^\bullet(P) = H^\bullet(X) \oplus H^{\bullet-2}(X) \oplus \cdots \oplus H^{\bullet-2n}(X)$

$D^b(P) = \langle p^* D(X),\ p^* D(X) \otimes \mathcal{O}_{P/X}(1),\ p^* D(X) \otimes \mathcal{O}_{P/X}(2), \ldots, p^* D(X) \otimes \mathcal{O}_{P/X}(n) \rangle$

"semi-orthogonal decomposition"

② if $f : \tilde{X} \to X$ is the blow-up along $Y \subset X$ smooth of codim n then

$H^\bullet(\tilde{X}) = H^\bullet(X) + H^{\bullet-2}(Y) + H^{\bullet-4}(Y) + \cdots + H^{\bullet-2n+2}(Y)$

$D(\tilde{X}) = \langle p^* D(X),\ n-1 \text{ copies of } D(Y) \rangle$

read Huybrechts Fourier-Mukai...

③ Whenever $H^\bullet(X)$ is related to $H^\bullet(Y)$

expect $D(X)$ related to $D(Y)$

example: $X = \cap\ 2$ quadrics in $\mathbb{P}^5$

w/ genus 2 curve C

$H^\bullet(X) = H^{\bullet-2}(C) + \mathbb{Z}$ in deg 0 and 6 (Reid)

$D(X) = \langle D(C),$ two copies of $D(pt) \rangle$ (Bondal-Orlov)

So if $D(X) \cong D(Y)$, what cohomological invariants are preserved?

- Hochschild homology $HH_i = R^i\Gamma(\mathcal{O}_\Delta \overset{L}{\otimes} \mathcal{O}_\Delta)$ in $X \times X$

if char $k = 0$ or $> \dim X$ then

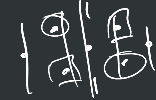
$HH_i = \bigoplus_{p-q=i} H^{p,q}(X)$ → over $\mathbb{C}$ $H^{p,q} = H^q(\Omega^p_X)$

→ over other fields, define it as $H^q(\Omega^p_X)$

if char $k = \dim X$, still ok (Antieau + Vezzosi '17)

but lose $H^{pq} = H^{qp}$

may fail if char $k < \dim X$ (Antieau, Bhatt, Mathew '19)

- if $\dim X \leq 2$ then $H^{p,q}(X) = H^{p,q}(Y)$ $\forall\, p,q$ 
  (Antieau + Bragg '19)

- $b_1 := \dim H^1_{\acute{e}t}(X, \mathbb{Q}_\ell)$ Popa + Schnell / char 0
  Honigs et al. / char p

  $\rightsquigarrow$ $H^{1,0}$ and $H^{0,1}$ are inv. in char 0, though not in char p.

  $\rightsquigarrow$ all $H^{p,q}$ are inv. in char 0 for 3-folds.

- Abuaf: in char 0 for $\dim X \leq 4$,
  $H^*(\mathcal{O}_X)$ as a ring is der. inv.

- Orlov conjectured that $D(X) \cong D(Y)$
  implies Chow motives $\otimes \mathbb{Q}$ are isomorphic.
  - $\rightsquigarrow$ point counts are equal / $\mathbb{F}_q$ (fails / $\mathbb{Q}$)
  - $\rightsquigarrow$ hodge #s are equal / $\mathbb{C}$ (fails / $\overline{\mathbb{F}}_p$)

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but π_1 is not invariant: Bak, Schnell using an example of Gross + Popescu.

$$\begin{array}{ccl} X & \xrightarrow[\text{res.}]{\text{small}} & Y = \cap\, 4 \text{ quadrics} \subset \mathbb{P}^7 \\ \downarrow & & \end{array}$$

fibered in Abelian surfaces: $X \to \mathbb{P}^1$

containing an ab. surface

$\rightsquigarrow$ Y has some nodes

$\mathcal{O}_{\mathbb{P}^7}(1)|_{\text{ab. surface}}$ is not a principal polarization, rather

$$0 \to \underbrace{\mathbb{Z}/8 \times \mathbb{Z}/8}_{=:G} \to A \to \hat{A} \to 0$$

in fact G acts freely on X and

$X/G \cong \overline{\mathrm{Pic}}^0(X/\mathbb{P}^1) =: M$ (dual ab. fibration)

Mukai: $D(A) \cong D(\hat{A})$

here: $D(X) \cong D(M)$

$\pi_1(X) = 0$ $\quad \pi_1(M) = G = \mathbb{Z}/8 \times \mathbb{Z}/8$ really μ_8

me 2013: $Br(X) = \mathbb{Z}/8 \times \mathbb{Z}/8$ and $Br(M) = 0$

Bragg's idea: $H^1(M, \mathbb{Z}/2) \neq 0$

if we were over $\overline{\mathbb{F}}_2$, Artin-Schreyer seq

$$0 \to \mathbb{Z}/2 \to \mathcal{O}_M \longrightarrow \mathcal{O}_M \to 0$$
$$f \longmapsto f^2 - f$$

would give $H^1(M, \mathbb{Z}/2) \hookrightarrow H^1(\mathcal{O}_M)$ so $H^{0,1}(M) \neq 0$

but $H^{0,1}(X) = 0$

tried Gross + Popescu's eqns over $\overline{\mathbb{F}}_2$

... doesn't work

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went back to their paper, found another example that works.

$$\mathbb{Z}/6 \times \mu_6 \circlearrowright X \longrightarrow Y = \cap \text{ 2 cubics in } \mathbb{P}^5 \qquad \text{over } \mathbb{C} \text{ or } \overline{\mathbb{F}}_3$$
$$\downarrow$$
$$\mathbb{P}^1$$

issues: ① $\mathbb{Z}_3 \times \mu_3 \subset \mathbb{Z}_6 \times \mu_6$ acts freely

but $\mathbb{Z}_2 \times \mu_2$ does not. $\qquad M = $ resolution of X/G

② Abelian fibration $\qquad \pi_1(M) = \mathbb{Z}/3 \times \mathbb{Z}/3$ over $\mathbb{C}$ ...

$X \to \mathbb{P}^1$ has reducible fibers

so $\overline{\mathrm{Pic}}^0(X/\mathbb{P}^1)$ is tricky

but we got through 'em.

Hodge #s of X or M / $\mathbb{C}$: $\qquad X/\overline{\mathbb{F}}_3$: $\qquad M/\overline{\mathbb{F}}_3$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 6 & & 0 & \\ 1 & & 6 & & 6 & & 1 \\ & 0 & & 6 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 7 & & 1 & \\ 1 & & \boxed{8} & & 8 & & 1 \\ & 1 & & 7 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 7 & & 0 & \\ 1 & & 6 & & 6 & & 1 \\ & 0 & & 7 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array}$$

$H^1(\mathcal{O})$

$H^0(\Omega^2_X) = H^0(T_X) = $ 1-dim'l

in char 0

$$0 \to \underline{\mathbb{Z}/3 \times \mathbb{Z}/3} \to \underline{\underline{\mathrm{Br}\, X}} \to \mathrm{Br}\, M \to 0$$