KAWAMATA BOUNDEDNESS FOR FANO THREEFOLDS AND THE GRADED RING DATABASE

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Abstract. We explain an effective Kawamata boundedness result for Mori–Fano 3-folds. In particular, we describe a list of 39,550 possible Hilbert series of semistable Mori–Fano 3-folds, with examples to explain its meaning, its relationship to known classifications and the wealth of more general Fano 3-folds it contains, as well as its application to the on-going classification of Fano 3-folds.

1. Introduction

The *Graded Ring Database*, or Grdb, is a database of information relating to polarised varieties and their graded rings. We are interested here in complex Fano 3-folds. The systematic study of all their graded rings together was initiated by Miles Reid; see [ABR02], which alongside [IF00, Alt98] was the starting point for the Grdb. The classification of Fano 3-folds remains a distant goal, even though a coarse answer is contained in the Grdb data if only we could distinguish the wheat from the chaff. This paper makes that statement, and its strengths and limitations, precise; the crucial points are summarised in §5.

Definition 1.1. A *Mori–Fano* 3*-fold* (sometimes called a $\mathbb{Q}$*-Fano* 3*-fold*) is a normal, complex, 3-dimensional projective variety X with $-K_X$ ample, Picard rank $\rho_X = 1$ and terminal, $\mathbb{Q}$-factorial singularities. The *genus* of X is $g_X := h^0(X, -K_X) - 2$.

Mori–Fano 3-folds are one of the possible end products of the Minimal Model Program in three dimensions [Mor88, (0.3.1)]. As yet, there is no classification, but it is known that there are only finitely many deformation families, with several hundred already described (see §4). The point of the Grdb is to put a concrete upper limit on the classification by providing a finite list of possible Hilbert series of Mori–Fano 3-folds. Although the list we derive is certainly too large, it does include all cases that exist (modulo one caveat; compare 2.6 and 2.9), and in fact it includes all Fano 3-folds we know far more generally.

There are many celebrated proofs of the boundedness of different classes of Fano varieties. The Grdb implements the Kawamata boundedness conditions [Kaw92], one of the earliest for 3-folds, and one which seems extraordinarily well suited to explicit classification, as we explain. The proof of [Kaw92, Prop 1] determines inequalities $-K_X^3 \le -\kappa K_X c_2(X)$ for certain values of κ, which in turn impose numerical constraints on the Hilbert series $P_X(t)$ of a Mori–Fano 3-fold X (polarised by $-K_X$; see §2.1).

A Fano 3-fold is said to be *semistable* if the sheaf $(\Omega^1_X)^{\star\star}$ is semistable. In this case, the inequality above holds with $\kappa = 3$. We derive a list $\mathcal{F}_{\mathrm{ss}}$ of rational functions which satisfy this additional condition.

Theorem 1.2. *Let X be a semistable Mori–Fano 3-fold. Then the Hilbert series $P_X(t)$ is one of the* 39,550 *rational functions on the list $\mathcal{F}_{\mathrm{ss}}$.*

In fact, relaxing $\kappa = 3$ to $\kappa = 4$ accommodates most of Kawamata's conditions even in the non-semistable case; see 2.6. By imposing only that weaker condition, we derive a larger list $\mathcal{F}_{\mathrm{MF}}$ of 52,646 rational functions that contains $\mathcal{F}_{\mathrm{ss}}$ but includes Hilbert series of possible non-semistable Fano 3-folds; see 2.9.

We refer to this pair of lists $\mathcal{F}_{\mathrm{ss}} \subset \mathcal{F}_{\mathrm{MF}}$ as *the Fano* 3*-fold database*. We adopt an abuse of language by saying that a Fano 3-fold X is in $\mathcal{F}_{\mathrm{ss}}$ to mean that $P_X(t) \in \mathcal{F}_{\mathrm{ss}}$. The Fano 3-fold database is most easily accessed online [BK09]; the raw data is available at [BK22], released under a CC0 licence [CC0], and code to generate it at [BK09].

We emphasise that many, perhaps most, of the rational functions on the list $\mathcal{F}_{\mathrm{MF}}$ cannot be realised as the Hilbert series of a Mori–Fano 3-fold; see §5.3 for a recapitulation of this point and §5.1 for several other possible confusions. Furthermore, we do not know a single example of a Mori–Fano 3-fold not in $\mathcal{F}_{\mathrm{ss}}$, though we know a small number of examples with canonical singularities such as the weighted projective space $\mathbb{P}(1, 1, 3, 5)$.

Genus

Codimension	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1		54	32	6	2	1																																		
2	1	45	29	6	2	1	1																																	
3		26	29	8	3	2	1	1																																
4	3	60	54	15	6	3	2	1	1																															
5	1	58	63	21	8	6	3	2	1	1																														
6	4	80	98	35	15	8	6	3	2	1	1																													
7		81	116	49	21	15	8	6	3	2	1	1																												
8	6	128	182	79	35	21	15	8	6	3	2	1	1																											
9	3	107	208	109	49	35	21	15	8	6	3	2	1	1																										
10	10	194	314	171	79	49	35	21	15	8	6	3	2	1	1																									
11	5	168	378	236	109	79	49	35	21	15	8	6	3	2	1	1																								
12	7	244	538	353	171	109	79	49	35	21	15	8	6	3	2	1	1																							
13	8	254	638	485	236	171	109	79	49	35	21	15	8	6	3	2	1	1																						
14	11	377	879	698	353	236	171	109	79	49	35	21	15	8	6	3	2	1	1																					
15	10	349	889	870	480	351	236	171	109	79	49	35	21	15	8	6	3	2	1	1																				
16	15	509	802	998	675	473	349	235	171	109	79	49	35	21	15	8	6	3	2	1	1																			
17	21	501	480	740	786	637	456	342	233	171	109	79	49	35	21	15	8	6	3	2	1	1																		
18	17	604	284	318	773	658	571	427	331	228	169	109	79	49	35	21	15	8	6	3	2	1	1																	
19	23	465	103	43	401	516	488	474	380	308	221	167	108	79	49	35	21	15	8	6	3	2	1	1																
20	31	220	44	4	106	159	267	299	345	308	272	204	160	106	79	49	35	21	15	8	6	3	2	1	1															
21	32	3	5		8	24	42	101	137	202	213	213	176	149	101	77	49	35	21	15	8	6	3	2	1	1														
22	27		1		1	1	3	7	28	43	88	117	141	135	128	94	75	48	35	21	15	8	6	3	2	1	1													
23	13							1	1	5	7	25	43	71	84	98	78	68	46	35	21	15	8	6	3	2	1	1												
24	10										1	1	5	10	23	36	59	54	57	41	33	21	15	8	6	3	2	1	1											
25	4													1	1	4	10	25	27	39	35	31	20	15	8	6	3	2	1	1										
26	2																1	1	7	9	20	23	24	18	15	8	6	3	2	1	1									
27																				1	1	6	10	15	13	13	8	6	3	2	1	1								
28																							1	2	4	8	11	7	6	3	2	1	1							
29																										1	2	6	5	6	3	2	1	1						
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FIGURE 1. Number of semistable Fano 3-fold Hilbert series in $\mathcal{F}_{ss}$, listed by genus and estimated minimal codimension.

We explain the proof of 1.2 in §2.4. This may be thought of as folklore, and similar in spirit to Prokhorov's degree bound computation [Pro07, (1.2)], but nevertheless the steps of turning Kawamata's boundedness theorem into a concrete list are important in light of the boundedness of Fano varieties more generally [KMMT00, Bir21]. The main results of this paper supplement 1.2 as follows:

(1) For each $P \in \mathcal{F}_{\text{ss}}$, we estimate the smallest anticanonical embedding that a Mori–Fano 3-fold X with Hilbert series $P_X = P$ could have (§3), and we use those to present $\mathcal{F}_{\text{ss}}$ as a geographical map (Figure 1). This geography, and its meaning as the basis for a programme of classification, is the main result of this paper.
(2) Most toric Fano 3-folds are not Mori–Fano 3-folds, yet almost all appear in $\mathcal{F}_{\text{ss}}$ (§4.5) and we plot those on the Fano geography (Figures 6, 7).
(3) We identify some known classifications within the Fano 3-fold database (§4) and compare with some well-known results (§3.3).
(4) We highlight possible misunderstandings of the Fano 3-fold database (§5.1, §5.3).
(5) The numbers of cases $\#\mathcal{F}_{\text{ss}} = 39{,}550$ and $\#\mathcal{F}_{\text{MF}} = 52{,}646$ arise from elementary combinatorial considerations (§2.1). This part of the statement is better thought of as a computational matter, and we provide computer code [BK09] that may be used to recreate the Fano 3-fold database.

We put some emphasis on toric Fano 3-folds throughout, in part to profit from simultaneous use of the Fano 3-fold database and the toric Fano classification in the Grdb. We indicate in §4.5 some different ways that this interdisciplinarity may yet be exploited.

The beauty and enduring interest of Fano classification lies in the individual 3-folds and deformation families we meet, rather than the bureaucracy of compartmentalising them. A map is only a map: the actual adventures happen out in the field, and the real value of the geography in Figure 1 is to identify hundreds of wonderful places to explore.

2. Building the Fano 3-fold database

We recall standard material related to the plurigenus formula and use that to assemble the data that proves 1.2 (compare [ABR02, §4]).

2.1. Fano 3-folds.

The right level of generality is the following.

Definition 2.1. A *Fano 3-fold with canonical singularities* is a normal, complex, 3-dimensional projective variety with canonical singularities and ample anticanonical class. We say 'Fano 3-fold' as an abbreviation for 'Fano 3-fold with canonical singularities'.

Any Fano 3-fold X comes with an intrinsic embedding $X \subset w\mathbb{P}$ in weighted projective space (up to automorphisms of $w\mathbb{P}$) as follows. The *graded ring* of X is

$$R(X, -K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X, -mK_X)$$

and $X \cong \operatorname{Proj} R(X, -K_X)$ by ampleness. Any choice $f_0, \dots, f_n$ of minimal homogeneous generators for $R(X, -K_X)$ has the same collection of weights $\{a_0, \dots, a_n\}$, where $a_i = \deg f_i$, and we may suppose $a_0 \le \cdots \le a_n$. (Note the conventional abuse of notation: $\{a_0, \dots, a_n\}$ is a list with possible repetitions, even though we use set notation. Thus, for example, $\{a_0, \dots, a_n\} \setminus \{a_3, a_7\}$ is the list obtained by removing one instance of each of a_3 and a_7, while leaving any other instances of the same numbers.) Thus any choice of minimal generating set determines an embedding

$$X \subset \mathbb{P}(a_0, \dots, a_n)$$

which we refer to as the *anticanonical embedding of* X. Note that this is not the same as the image $\Phi_{-K_X}(X)$ of X by the linear system $|-K_X|$, unless $-K_X$ is very ample, which is frequently not the case.

The *Hilbert series* of a Fano 3-fold X is the formal power series

$$P_X = P_X(t) = \sum_{m\in\mathbb{N}} h^0(X, -mK_X)t^m$$

which is the Hilbert series of $R(X, -K_X)$. A simple but important theme throughout this paper is that the Hilbert series P_X of a Fano 3-fold X does not determine the weights a_i of its anticanonical embedding.

2.2. **Numerical data of a Fano 3-fold.** Any Fano 3-fold has a *basket of singularities* $\mathcal{B}$, which is a collection of terminal quotient singularities $\frac{1}{r}(1,a,-a)$ (possibly including repeats, where again we use set notation on this understanding). Following [Rei87, §§8–10], in general $\mathcal{B}$ is derived locally from the singularities of X by crepant blowup and Q-smoothing. When X has at worst terminal singularities and lies in weighted projective space as a quasismooth variety, then X is an orbifold and $\mathcal{B}$ is exactly the collection of singularities of X.

The *genus* $g \geq -2$ of X is defined by $h^0(X, -K_X) = g+2$. The genus g and basket $\mathcal{B}$ together determine the degree by the formula

$$-K_X^3 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r} \tag{2.A}$$

where for each element $\frac{1}{r}(1,a,-a)$ of the basket $\mathcal{B}$, we define b by $ab \equiv 1 \mod r$.

Although they are not used for the initial construction of the GRDB, there are various different divisorial indices defined for a Fano 3-fold X that we consider later.

Definition 2.2. The *Gorenstein index* or *singularity index* of X is

$$i_X = \min\{n \in \mathbb{Z}_{>0} \mid nK_X \text{ is Cartier}\}.$$

The *Fano index* is $f_X = r/i_X$ where $r > 0$ is the largest integer for which $-i_X K_X \equiv rA$ for a Cartier divisor A. There are further types of Fano index:

$$q_X = \max\{q > 0 \mid -K_X \sim qA, \text{ where } S \text{ is a } \mathbb{Q}\text{-Cartier Weil divisor}\}$$
$$q_{\mathbb{Q}X} = \max\{q > 0 \mid -K_X \sim_{\mathbb{Q}} qA, \text{ where } S \text{ is a } \mathbb{Q}\text{-Cartier Weil divisor}\}.$$

These are all natural generalisations of the divisibility of the anticanonical divisor of a smooth Fano 3-fold in its Picard group. For a Fano 3-fold, both q and $q_{\mathbb{Q}}$ are positive integers, and they are equal if $\mathrm{Cl}(X)$ has no torsion [Pro10], while f_X may be strictly rational.

2.3. **Effective Kawamata boundedness for Mori–Fano 3-folds.** We construct a set of genus–basket pairs $(g, \mathcal{B})$ that satisfy the constraints of [Kaw92].

2.3.1. *Possible baskets.* Controlling the possible baskets of Fano 3-folds can be done much more generally than the Mori–Fano case.

Theorem 2.3 ([Kaw86, Lemma 2.2, 2.3], [Rei87, (10.3)])**.** *Let X be a projective 3-fold with canonical singularities. Then*

$$24\chi(\mathcal{O}_X) = -K_X c_2(X) + \sum_{\mathcal{B}} r - \frac{1}{r} \tag{2.B}$$

where the sum is over the singularities $\frac{1}{r}(1,a,-a)$ of the basket $\mathcal{B}$ of X.

When X is a Fano 3-fold, $\chi(\mathcal{O}_X) = 1$ and (2.B) simplifies.

Lemma 2.4. *There is a list $\mathbb{B}$ of 8314 baskets with the following property: if X is any Fano 3-fold with canonical singularities which satisfies $-K_X c_2(X) > 0$, then the basket $\mathcal{B}$ of X is in $\mathbb{B}$.*

Proof. The condition $-K_X c_2(X) > 0$ together with (2.B) implies that $\sum r - (1/r) < 24$, where the sum is taken over the basket $\mathcal{B}$. This implies first that $\#\mathcal{B} \leq 15$, since $r - 1/r \geq 3/2$ for each $\frac{1}{r}(1,a,-a) \in \mathcal{B}$, and further that each such $r \leq 24$. Therefore there are only finitely many possible collections of indices r appearing in $\mathcal{B}$.

The only terminal quotient singularity with $r = 2$ is $\frac{1}{2}(1,1,1)$, and for each index $r > 2$, there are $\phi(r)/2$ terminal quotient singularities, namely $\frac{1}{r}(1,a,-a)$ for $1 \leq a < r/2$ coprime to r. Thus for each index r that appears in $\mathcal{B}$, there are only finitely many singularities of index r that may occur. Enumerating all possible baskets satisfying these conditions (by computer, for example) gives 8314 cases. ◻

Remark 2.5. The condition $-K_X c_2(X) > 0$ holds for Mori–Fano 3-folds by 2.6. Far more generally, any weak Fano 3-fold (that is, $-K_X$ is only required to be nef and big) with terminal singularities satisfies $-K_X c_2(X) \geq 0$ by [KMMT00, 1.2(1)]. It is easy to check that relaxing the inequality provides (coincidentally) 24 additional cases with $\sum r - \frac{1}{r} = 24$, such as $16 \times \frac{1}{2}(1,1,1)$ and $5 \times \frac{1}{5}(1,2,3)$. Thus the scope for there to be Fano 3-folds not lying in $\mathcal{F}_{\mathrm{MF}}$ is less about the possible baskets and more about the maximum permitted genus for each basket, which we come to next.

2.3.2. *Genus bounds.* Controlling the possible values for the genus g_X for each basket, uses the full power of [Kaw92], and so prima facie applies only in the Mori–Fano case.

Theorem 2.6 (Kawamata [Kaw92])**.** *Let X be a Mori–Fano 3-fold. Then*

$$-K_X^3 \le \kappa(-K_X c_2(X)) \tag{2.C}$$

for some real number $\kappa > 0$. In particular, $-K_X c_2(X) > 0$.

If X is semistable, then the formula (2.E) holds with $\kappa = 3$. If $(\Omega_X^1)^{\star\star}$ has a rank 2 maximal destabilising subsheaf, then the formula holds with $\kappa = 4$. If $(\Omega_X^1)^{\star\star}$ has a rank 1 maximal destabilising subsheaf, then it holds with possibly larger $\kappa > 0$.

Corollary 2.7. *Let X be a Mori–Fano 3-fold with basket $\mathcal{B}$ and genus g. Then $g_{\min} \le g \le g_{\max}$ where*

$$g_{\min} = \max\left\{-2, \left\lfloor \frac{1}{2}\left(2 - \sum_{\mathcal{B}} \frac{b(r-b)}{r}\right)\right\rfloor + 1\right\} \tag{2.D}$$

and

$$g_{\max} = \left\lfloor \frac{1}{2}\left(2 - \sum_{\mathcal{B}} \frac{b(r-b)}{r} + \kappa\left(24 - \sum_{\mathcal{B}} r - \frac{1}{r}\right)\right)\right\rfloor \tag{2.E}$$

where each sum is over $\frac{1}{r}(1, a, -a) \in \mathcal{B}$ and b is defined by $ab \equiv 1 \mod r$, and $\kappa > 0$ in (2.E) is as determined in 2.6.

Proof. The lower bound is simply the condition that $-K_X^3 > 0$ in (2.A). For the upper bound, substituting (2.A) and (2.B) into (2.C) gives

$$2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r} \le \kappa\left(24\chi(\mathcal{O}_X) - \sum_{\mathcal{B}} r - \frac{1}{r}\right)$$

and the upper bound follows. □

2.4. **Proof of 1.2.** The Hilbert series P_X is equivalent to the data of genus–basket pair $(g, \mathcal{B})$ by the following Fletcher–Reid plurigenus formula together with (2.A) and the independence of basket contributions [Fle89, 4.2].

Theorem 2.8 ([Fle87, Theorem 2.5], [Rei87, (10.3)])**.** *Let X be a Fano 3-fold with canonical singularities and basket $\mathcal{B}$. Then*

$$P_X(t) = \frac{1+t}{(1-t)^2} - \frac{t(1+t)}{(1-t)^4}\frac{K_X^3}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r - \overline{bi})t^i}{2r}, \tag{2.F}$$

where for each element $\frac{1}{r}(1, a, -a)$ of the basket $\mathcal{B}$, we define b by $ab \equiv 1 \mod r$, and $\overline{c} \in \{0, 1, \dots, r-1\}$ denotes the least residue modulo r.

Given values for g and $\mathcal{B}$, the formulas (2.A) and (2.F) determine a rational function, denoted $P_{g,\mathcal{B}}$, that is the Hilbert series of any Fano 3-fold with these genus and basket.

Proposition 2.9. *There are 39,550 genus–basket pairs $(g, \mathcal{B})$ that could be the genus and basket of a semistable Mori–Fano 3-fold. Relaxing the semistability condition of 2.6 from $\kappa = 3$ to $\kappa = 4$ gives 52,646 such pairs.*

Proof. Lemma 2.4 provides exactly 8314 possible baskets. Of these, (2.E) with $\kappa = 4$ calculates $g_{\max} \ge -2$ in 7683 cases, or 7492 cases with $\kappa = 3$. Assembling pairs $(g, \mathcal{B})$ with $\mathcal{B}$ one of these baskets and $g_{\min} \le g \le g_{\max}$ bounded by 2.7 gives 52,654 cases, or 39,558 with the semistable condition.

By 2.8 and (2.A), each genus–basket pair $(g, \mathcal{B})$ determines a formal power series that is the Hilbert series of any X with genus g and basket $\mathcal{B}$. Eight of the resulting series, each corresponding to a semistable pair, have expansions starting

$$1 + t + t^2 + \cdots + t^n + O(t^{n+2}),$$

with either $n = 2$ or $n = 4$, and the t^{n+1} term having coefficient zero. Such series cannot be the Hilbert series of a reduced scheme, since powers of the necessary generator, x say, of degree 1 generate each graded piece up to degree n, but then $x^{n+1} = 0$. We discard these eight series at this stage, to leave 39,550 series in $\mathcal{F}_{\mathrm{ss}}$ and 52,646 series in $\mathcal{F}_{\mathrm{MF}}$. □

Computer code to enumerate the Hilbert series of 2.9, either in the Go-language [Goo17] or independently for the Magma system [BCP97], is available at [BK09].

3. The geography of Fano 3-folds

For each rational function $P \in \mathcal{F}_{\mathrm{MF}}$, the Grdb gives a collection of 'weights' $a_0, \dots, a_n$ so that the product

$$P \cdot \prod_{i=0}^{n} (1 - t^{a_i}) = 1 - \sum_{d_i} t^{d_i} + \sum_{e_i} t^{e_i} - \cdots \pm t^k. \tag{3.A}$$

is a polynomial. This polynomial is called the *Hilbert numerator of* P_X, which of course depends on the choice of weights a_i. The point is that if there really is a Fano 3-fold X embedded as

$$X \subset \mathbb{P}^n(a_0, \dots, a_n) \tag{3.B}$$

then the expression (3.A) is related to the equation degrees d_i, syzygy degrees e_i and adjunction number k of the defining equations; see [Rei02, 3.6]. To be suggestive and provocative, Grdb presents each genus–basket pair $(g, \mathcal{B})$ in the form (3.B), for weights chosen to suit the corresponding series $P_{g,\mathcal{B}}$. One must be aware that there could be many different apparently 'good' choices of weights, and it is important to understand how the Grdb weights are chosen, since, as we explain in §5.1, there are many traps to fall into when interpreting them.

The process used to assign weights in the Grdb is inductive. The base of the induction is the known classification of weighted complete intersections, which we describe next.

3.1. **Hilbert series in low codimension.**

3.1.1. *The famous 95 and Chens' result.* The famous 95 weighted hypersurfaces of Reid [Rei80, (4.5)], Johnson–Kollàr [JK01, BK16], and others realise the codimension 1 (top) row of Figure 1. This classification of hypersurfaces is well established: if a Mori–Fano 3-fold is anticanonically embedded as a hypersurface in weighted projective space, then it is in one of the 95 families. The converse is not true: a variety may have the Hilbert series of one of the 95 without being a hypersurface, such as a non-general complete intersection $X_{2,4} \subset \mathbb{P}(1^5, 2)$, or similar degenerations in [Bro07, Table 2].

In similar vein, Iano-Fletcher, [IF00, (16.7) Table 6] lists 85 families of Fano 3-folds in codimension 2, realising the codimension 2 (second) row of Figure 1. Chen–Chen–Chen [CCC10] prove that this list is complete in the following sense: if $X \subset \mathbb{P}(a_0, \dots, a_5)$ is a codimension 2 complete intersection Mori–Fano 3-fold, then either it is in one of Iano-Fletcher's 85 families, or it is a degeneration of one of the famous 95 (or has a quasi-linear equation). Again the converse is not true: for example, [Bro06, Table 3] lists degenerations of 13 of the 85 families that lie in codimension 3 (described as K3 surfaces, but each extends to a Fano 3-fold with an additional variable of degree 1).

3.1.2. *Identifying low codimension Hilbert series.* Suppose $X \subset \mathbb{P}(1^{m_1}, 2^{m_2}, \dots)$ is a variety in a projectively normal embedding that is nondegenerate, in the sense that none of its defining equations is quasi-linear. Such X has a Hilbert series $P_X(t) = \sum_{i \geq 0} n_i t^i$, where $n_i = h^0(X, \mathcal{O}_X(i))$. In general, knowing $P_X(t)$ alone is not enough to determine the m_j, but it can provide estimates, even without information about the equations of X.

Suppose given a series $P_0 = 1 + n_1 t + n_2 t^2 + \cdots$, assumed to be the Hilbert series of X as above. First $m_1 = n_1$ by the nondegeneracy assumption. Consider $P_1 = (1 - t)^{m_1} P_0 = 1 + n_2' t^2 + \cdots$. If $n_2' \geq 0$, then there are necessarily at least that many variables in weight 2, so set $m_2 = n_2'$ and write $P_2 = (1 - t^2)^{m_2} P_1 = 1 + n_3'' t^3 + \cdots$. If now $n_3'' \geq 0$, then there are necessarily at least that many variables in weight 3, so set $m_3 = n_3''$ and consider $P_3 = (1 - t^3)^{m_3} P_2$. Necessarily at some stage $n_{r+1}^{(r)} < 0$, and the game ends: we can no longer conclude there are necessarily additional generators, and indeed there must be at least $-n_{r+1}^{(r)}$ relations of weight i.

If the result is to be a nondegenerate complete intersection, then the degrees of variables detected by this process are inevitably among those of any minimal generating set of the graded coordinate ring of X: if at the ith stage we had included an additional variable of weight i, that would have necessitated an equation of weight i, which in a nondegenerate complete intersection would eliminate the additional variable. If the game has continued far enough that the numerator $(1 - t^r)^{m_r} \cdots (1 - t)^{m_1} P_0$ with respect to the weights discovered so far is a polynomial, then we may now attempt to construct $X \subset \mathbb{P}(1^{m_1}, 2^{m_2}, \dots, r^{m_r})$. If we are lucky, we may construct such X with $P_X = P_0$ (and in particular, therefore, with no equations in weights $\leq r$) and check that it has whichever properties – irreducible, quasismooth, Fano, and so on – that we intended.

(In passing, note a simple example where this graded ring game needs a little thought. The genus 5 hyperelliptic curve $C_{2,6} \subset \mathbb{P}(1^3, 3)$ has Hilbert series $P = 1 + 3t + 5t^2 + 8t^3 + \cdots$, so the first step is to consider $(1-t)^3 P = 1 - t^2 + t^3 - t^5$. The naive game is complete, but clearly there is no variety defined by a single quadric in $\mathbb{P}^2$ with a linear syzygy. Speculatively looking ahead for the next positive coefficient suggests considering a variable of weight 3, which of course recovers the numerical data of C as $(1-t^3)(1-t)^3 P = 1 - t^2 - t^6 + t^8$. In practice, this phenomenon is rare, and when it does arise for complete intersections the solution is as simple as this example.)

Although as an algorithmic process this seems to give rather a lot away, when applied to the series $P_{g,\mathcal{B}} \in \mathcal{F}_{\rm ss}$ it recovers many known Fano 3-folds at once.

3.1.3. *Polarising baskets.* We also add weights to ensure there are global generators to realise the singularities of the basket correctly. For example, the case $g = 2$, $\mathcal{B} = \left\{\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)\right\}$ determines a series $P(t)$ that satisfies

$$(1-t)^4(1-t^2)^2 P(t) = 1 - 3s^4 + 3s^6 - 2s^7 + 3s^9 - 3s^{10} + \cdots,$$

suggesting that generators in degrees $1,1,1,1,2,2$ to start with, but does not say what other generators may be necessary. But of course there must be some ambient orbifold locus with stabiliser $\mathbb{Z}/3$, to allow for the index 3 orbifold point. That could be achieved with a generator in degree 3, or a combination of generators whose degree have 3 as their greatest common divisor. Here the simplest thing works:

$$(1-t)^4(1-t^2)^2(1-t^3) P(t) = 1 - t^3 - 3t^4 + 3t^6 + t^7 - t^{10}$$

suggests a variety $X \subset \mathbb{P}(1,1,1,1,2,2,3)$ in codimension 3 defined by five Pfaffians of degrees 3, 4, 4, 4 and 5. (Notice that the equation of degree 5 is masked in the Hilbert numerator by a syzygy of degree 5, the Hilbert numerator is 'really' $1 - \cdots - 3t^4 - t^5 + t^5 + \cdots - t^{10}$. Knowing that codimension 3 Gorenstein ideals have an odd number of generators defined as Pfaffians is extra information that comes from the Buchsbaum–Eisenbud theorem. It is easy to check that such a Fano 3-fold really exists.)

Other ways to introduce index 3 points, such as including weights 6 and 9, may also work, but result in higher codimension. In high codimension more complicated combinations such as these are sometimes used. The point is not to add weights of degrees smaller than the minimum equation degree to avoid imposing relations among the minimal generators that are not implied by the numerics alone. When the basket has several singularities, this check works in descending order of index, as new high-degree variables may polarise lower-degree singularities.

At this point, for each Hilbert series we have identified some simple low weights that are enough to generate the ring in low degree and to polarise the singularities. This already determines the weights used in GRDB in codimension ≤ 3, though is not always correct in high codimension. We fix this next.

3.2. **Numerical unprojection ansatz and weights.** Type I Gorenstein unprojection [PR04,KM83] is a technique that takes as input a pair of Gorenstein schemes $D \subset X$, with $D \subset X$ of codimension 1, and returns a new Gorenstein scheme Y. In applications to projective geometry, it often corresponds to a birational contraction of D to a point of Y, and that is how we wish to apply it.

3.2.1. *The Type I ansatz.* We describe a model case of Kustin–Miller unprojection following Papadakis–Reid [PR04, 2.4–9]. Consider the following hypothetical input-output process:

Input: Let $X \subset \mathbb{P}(a_0, \ldots, a_n)$ be a Fano 3-fold with terminal singularities in its anticanonical embedding with basket $\mathcal{B}_X$. Suppose $D \subset X$ for some coordinate plane $D = \mathbb{P}(a_i, a_j, a_k)$ with weights $(a_i, a_j, a_k) = (1, a, b)$ and $\gcd(a, b) = 1$, and suppose further that X is quasismooth away from finitely many nodes $\Sigma \subset D$.

Output: A quasismooth Fano 3-fold $Y \subset \mathbb{P}(a_0, \ldots, a_n, r)$ in its anticanonical embedding, with $r = a + b$, such that:

(1) Y contains the point $P = (0 : \ldots : 0 : 1)$ and $P \in X$ is a terminal quotient singularity $\frac{1}{r}(1, a, b)$.
(2) Y has the same genus as X. Furthermore, if $a + b + c > d$ for every $c, d \in \{a_0, \ldots, a_n\} \setminus \{a_i, a_j, a_k\}$, then the equations of Y have no quasi-linear terms.
(3) The Gorenstein projection from $P \in Y$ is a birational map $Y \dashrightarrow X$ that factorises into birational morphisms as follows:

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

where $Z \to Y$ is the contraction of the birational transform $D \subset Z$ to $P \in Y$ (which is the Kawamata blowup of $P \in Y$, viewed from Y), and $Z \to X$ is the small D-ample resolution of the nodes of X (which is the contraction of finitely many flopping curves, viewed from Z).

(4) The basket $\mathcal{B}_Y$ of Y satisfies

$$\mathcal{B}_X \cup \left\{ \frac{1}{r}(1,a,b) \right\} = \mathcal{B}_Y \cup \left\{ \frac{1}{a}(1,b,-b), \frac{1}{b}(1,a,-a) \right\} \tag{3.C}$$

where $\frac{1}{a}(1,b,-b)$ is omitted if $a=1$, and analogously if $b=1$.

(5) The Hilbert series of Y is

$$P_Y = P_X + \frac{t^{a+b+1}}{(1-t)(1-t^a)(1-t^b)(1-t^r)}.$$

In many cases this process is a theorem; see [BKR12, 3.2] for example. Indeed the setup $D \subset X$ satisfies the conditions for Kustin–Miller unprojection [PR16, 2.4], giving a new variable s of degree $r = k_X - k_D = -1-(-1-a-b) = a+b$ and additional equations involving s of the form $sf_i = g_i$, where f_i form a basis of the ideal I_D in the coordinate ring of X. One can see using a free resolution of the coordinate ring $\mathbb{C}[Y]$ over $\mathbb{C}[\mathbb{P}(a_0,\dots,r)]$ ([Pap04]) that $\mathcal{O}(-K_Y) = \mathcal{O}(1)$, and the numerics of (1), (2), (4), and (5) follow (cf. [PR16, 2.7–9]). Then given Y, the Kawamata blowup of $P \in Y$ is a weak Fano, and it has an anticanonical model $Z \to X'$. More complicated situations can arise – see (4.B), where $Z \to X$ makes both a crepant divisorial contraction and a disjoint flopping contraction – but since the assumption of nodes here already establishes that contracting the flopping curves result in a Fano, there can be no further contraction in the given situation.

However, here we do not use the setup above as a theorem to be applied, rather we turn it around to act as an ansatz, as follows.

Ansatz 3.1 (Type I unprojection)**.** Suppose that a genus–basket pair $(g, \mathcal{B}_Y) \in \mathcal{F}_{\mathrm{MF}}$ is not among the $95+85$ cases assigned weights in codimension 1 or 2 by §3.1.2, and that $(g,\mathcal{B}_X)$ is another genus–basket pair (with matching genus) which satisfies (3.C) for suitably coprime $r = a+b$. If the weights $(a_0,\dots,a_n)$ of X listed in GRDB contain $(1,a,b)$ as a sublist, then we insist that the weights of Y in GRDB are $(a_0,\dots,a_n,r)$.

This ansatz works as an inductive procedure from low to high codimension, taking the codimension 1 and 2 complete intersections as given, and it is simple to arrange in any particular case. The main point is the empirical result that this operation is well defined over the whole Fano 3-fold database; the proof is simply a (computer) consistency check across the Fano 3-fold database.

Lemma 3.2 (Type I consistency)**.** *Whenever some pair $(g,\mathcal{B}_Y)$ admits the Type I unprojection relation 3.1 to different pairs $(g,\mathcal{B}_{X_1})$ and $(g,\mathcal{B}_{X_2})$, then the weights for $(g,\mathcal{B}_Y)$ determined by 3.1 are independent of which X_i pair is used.*

To give some idea of the potency of this result, only 1087 of the 39,370 genus–basket pairs not among the $95+85$ cases in codimension ≤ 2 do not satisfy the Type I projection numerics for $(g,\mathcal{B}_Y)$ in 3.1. But to be clear: the claim is not that for each of these $(g,\mathcal{B}_Y)$ pairs we may find a particular $D \subset X$ that satisfies the conditions specified as input to a Type I unprojection above. The claim is merely that the numerics of the weights are consistent with its existence. Thus there is no promise that we will be able to make unprojections in accordance with 3.1 in every case, thereby realising most of the Hilbert series (but compare §4.7 for an attempt to realise this stronger claim).

Remark 3.3. In fact, more is true. There is a class of more complicated Gorenstein projections, referred to as Type II_n for $n \geq 1$; see [Pap08, Tay]. These may also be used to describe weights for genus–basket pairs based on the same relation (3.C) but in the case one of the polarising weights $c \in \{1,a,b\}$ does not lie among the weights of X, but $(n+1)c$ does, for minimal $n \geq 1$ (and disjoint from the other polarising weights). In this case, the unprojection adjoins $n+1$ variables of weights $r, r+c, \dots, r+nc$. For example, a particular $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ would arise by Type II_2 projection from $\frac{1}{7}(1,2,5) \in Y \subset \mathbb{P}(2,3,4,5,6,7^2,8,9)$, if the latter exists, since $\{(n+1)\times 1, 2, 5\}$ is a sublist of the weights of X for $n=2$, but not for smaller $n \geq 1$.

Once more, the weights that this projection comparison process determine are consistent with all possible Type II_n projections, and also with all those coming from Type I projections; compare [Bro07, 3.4]. For example, 735 of the 52,646 genus–basket pairs do not admit a numerical Type I projection, but do admit a numerical Type II_1 projection (and, in fact, all lie in $\mathcal{F}_{\mathrm{ss}}$); a further 159 admit a Type II_2, then a further 68 with Type II_3, 24 with Type II_4 and so on with diminishing returns.

3.3. **Numerical corollaries.** The crude classification by 1.2 already contains enough information to provide approximations to various strong and sharp theorems by elementary and easy means. For example, recall Prokhorov's sharp bound on the degree.

Theorem 3.4 (Prokhorov [Pro07])**.** *If X is a Mori–Fano 3-fold which is not Gorenstein, then $-K_X^3 \leq 62\frac{1}{2}$, and this bound is realised only by $X = \mathbb{P}(1^3, 2)$.*

In the semistable case, Theorem 1.2 recovers the weaker bound $-K_X^3 \leq 72$ at once, and in conjunction with [Kar09, Kar15], this improves to $-K_X^3 \leq 66\frac{1}{2}$. The next outstanding case is $(g, \mathcal{B}) = (34, \{\frac{1}{2}(1,1,1)\})$, which we see (§4.2.1) is populated by the blowup $\mathrm{Bl}_p\,\mathbb{P}(1^2, 3, 5)$ at a smooth point P, and which Prokhorov's stronger geometric Sarkisov methods show cannot be realised by a Mori–Fano 3-fold.

Another example is the following result of [CCC10], proving [IF00, Conjecture 18.19(2)], which bounds the codimension of Fano complete intersections.

Corollary 3.5 ([CCC10, Theorem 1])**.** *If X is a Mori–Fano 3-fold whose anticanonical embedding $X \subset w\mathbb{P}^n$ is a complete intersection in weighted projective space, then $n \leq 6$.*

The proof is difficult and subtle, but in the semistable case this follows again from 1.2 by (computer-aided) inspection of the numerators of all Hilbert series in the database. Indeed, for a complete intersection, the process of determining the weights and numerator (§3.1.2) is well defined. Conversely, given weights, the numerator determines a minimal set of degrees for equations. In most cases, this immediately rules out a complete intersection, as there are too many equations – and adding additional weights does not alter that. Of the remaining, there are cases of apparent complete intersections, but by equations whose degrees are too small to accommodate the high-degree variables: therefore to be terminal there must be further equations, and again complete intersection is ruled out.

4. Populating the Fano 3-fold database

We locate some of the established classifications of particular classes of Fano 3-folds within the Fano 3-fold database $\mathcal{F}_{\mathrm{ss}} \subset \mathcal{F}_{\mathrm{MF}}$. The point is to gain some understanding of the accuracy of the database for the classification of Mori–Fano 3-folds, and to identify where the boundaries of our knowledge and the next questions lie. Known results suggest that Mori–Fano 3-folds may be clustered towards the top left-hand corner of Figure 1. We know many elements of $\mathcal{F}_{\mathrm{ss}}$ which have no matching Mori–Fano 3-fold (see §4.1,4.3).

More generally, Figure 1, which sketches only $\mathcal{F}_{\mathrm{ss}}$, seems to serve as a first guide to the classification of other classes of Fano 3-fold, and so exactly the same questions arise for Fano 3-folds with higher Picard rank, or with canonical singularities, and so on. We only know a single element of $\mathcal{F}_{\mathrm{ss}}$ for which it is proven that there is no matching Fano 3-fold with canonical singularities (see §4.2). We do know a few examples whose Hilbert series do not appear in $\mathcal{F}_{\mathrm{ss}}$: $X = \mathbb{P}(1^2, 3, 5)$ has isolated canonical singularities, with anticanonical embedding $X \subset \mathbb{P}(1^{36}, 2, 3)$, and is a perfectly respectable Fano 3-fold with $P_X \in \mathcal{F}_{\mathrm{MF}}$, but a glance at the $g = 34$ column of Figure 1 shows $P_X \notin \mathcal{F}_{\mathrm{ss}}$.

4.1. **Smooth Fano 3-folds.** The celebrated classification of 105 families of smooth Fano 3-folds [Isk78, Isk78, MM82], listed in [IP99, Table 12.2] and online at [Bel19], lies along the leading diagonal of Figure 1 in a fairly complicated way, as we indicate in Figure 2.

Each family listed in Figure 2 appears in the Grdb in their familiar anti-canonical model. Generalising to Gorenstein terminal Fano 3-folds does not increase the number of deformation families: by [Nam97], any Gorenstein terminal Fano 3-fold may be smoothed, so it appears in Figure 2, and furthermore by [JR11] the Picard rank does not change on smoothing (indeed [JR11, §2] uses this to determine smoothing families). In contrast, there are Fano 3-folds with canonical Gorenstein singularities that realise other families in the leading diagonal of Figure 1; we discuss this further in §4.2.

Figure 2 lists every deformation family of smooth Fano 3-folds using the numbering convention of [Bel19] (which adapts [IP99, MM82]). Since for a smooth Fano 3-fold X the genus g_X determines $-K_X^3$ and P_X, each Hilbert series may be common to many families, and it is useful to list families by the pair of invariants (g_X, ρ_X), using notation ρ-n to denote the n-th family of 3-folds with Picard rank ρ. Each row lists all families of a given genus g, and so one may imagine this table lying along the leading diagonal of Figure 1, with each row listing all families that correspond to a single entry in the Fano 3-fold database. The columns specify the Picard rank ρ, and are labelled ρ-n, where the values of n are the entries of the table and indicate the nth family of Picard rank ρ.

g	1-n		2-n				3-n				4-n	ρ-n	$\#\mathcal{T}$
2	1												0
3	2				1_{11}								1
4	3					2						10-1	7
5	4	11			3_{12}								
6	5		4^*										54
7	6				5_{13}	6				1		9-1	135
8	7			$7^\dagger$		8				2			207
9	8	12			10_{14}	9							314
10	9				11_{13}		3^*		4_{18}			8-1	373
11			12^*	$13^\dagger$	14_{15}		5^*						416
12	10		15^*		16_{14}				$6_{25,33}$				413
13		13	$17^{*\dagger}$			18	$7^*_{32}, 8^*_{24}$				1^*	7-1	413
14			19^*_{14}		20_{15}				$9_{36}, 10_{29}$		13^*		348
15				$21^\dagger$			$12^*_{27,33}$	$11^{*\dagger}_{25}$			2_{31}	5-1	344
16			22^*_{15}	$23^\dagger$		24			13_{32}		$3^*_{17,28}$	6-1	274
17		14	25^*				$15^*_{29,31}$	$14^\dagger_{36}$			$4^{30}_{18/9}, 5^{31}_{21/8}$		234
18				$26^\dagger_{15}$				$16^\dagger_{27,32}$			6^*_{25}		179
19							17^*		$18_{29,30/3}$		$7_{24,28}$	5-2/3	151
20			27^*				21^*	$19^\dagger$	$20_{31/2}$		8^*_{31}		117
21		15	28^*	$29^\dagger$			22^*_{36}				$9_{25/6/8,30}$		87
22							24^*_{32}	$23^\dagger_{30/1}$			10^*_{28}		66
23									25_{33}		$11_{28,31}$		40
24			30^*	$31^\dagger$				$26^{*\dagger}$			12_{30}		42
25						32	28^*			27			27
26								$29^\dagger, 30^\dagger_{33}$					18
27										31			8
28		16	33^*			34							13
29			35^*										9
30													4
31													2
32						36							2
33		17											5
			${}^*_{17}\mathbb{P}^3$	${}^\dagger_{16}X_2$			${}^*_{34}\mathbb{P}^1\times\mathbb{P}^2$	${}^\dagger_{35}\widehat{\mathbb{P}^3}$			${}^*_{27}(\mathbb{P}^1)^3$		

FIGURE 2. The 105 families of smooth Fano 3-folds, listed as ρ-n for the nth variety of Picard rank ρ, with a row for each genus $g = 2, \ldots, 33$.

Many of these Fano 3-folds with $\rho \geq 2$ are constructed by extremal extractions from other smooth Fano 3-folds. The table includes some of these extremal divisorial contractions by writing entries ρ-n_m, abbreviated to n_m in the table, to indicate a map from members of family ρ-n to members of $(\rho-1)$-m. Some codomains are very common, and we indicate these by the following special notation:

1. 2-$n^* \equiv$ 2-n_{17} means map to 1-17 $= \mathbb{P}^3$
2. 2-$n^\dagger \equiv$ 2-n_{16} means map to 1-16 $= X_2 \subset \mathbb{P}^4$
3. 3-$n^* \equiv$ 3-n_{34} means map to 2-34 $= \mathbb{P}^1 \times \mathbb{P}^2$
4. 3-$n^\dagger \equiv$ 3-n_{35} means map to 2-35 $= \mathrm{Bl}_P\, \mathbb{P}^3$
5. 4-$n^* \equiv$ 4-n_{27} means map to 3-27 $= \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

The columns are arranged to indicate some of this information. Fano manifolds X with $\rho_X = 1$ and Fano index $f_X = 1$, are on the left-hand side of column 1-n, with $f_X \geq 2$ on the right-hand side. Columns $\rho = 2, 3$ are arranged to give a brief indication of the extremal contraction data, with varieties admitting the common morphisms listed on the left-hand side of each column, with those admitting only other morphisms to the right of centre, and those with no birational contractions to another smooth Fano 3-fold down the right-hand side; most of these are products or double covers, and their extremal rays are Mori fibrations. The last column lists all $\rho = 5, \ldots, 10$ together, as these cases are sparse. (Column $\#\mathcal{T}$ lists the number of Gorenstein toric models by genus; see §4.5.)

4.2. **Gorenstein Fano 3-folds.** When X is a Fano 3-fold with canonical singularities that has $-K_X$ Cartier, then its singularities are Gorenstein and its basket is empty. Therefore its Hilbert series P_X behaves as though X has no singularities at all, and so again lies on the leading diagonal of Figure 1.

There is no classification of Fano 3-folds with Gorenstein canonical singularities (though [CPS05] settle the hyperelliptic and trigonal cases), but there are precise results for extreme cases, which we describe now in relation to the GRDB geography. Recall that at the pointy right-hand end of Figure 1 the number of Hilbert series in $\mathcal{F}_{\mathrm{ss}}$ by genus and codimension is:

Codim \ Genus	33	34	35	36	37
31	1				
32	1	1			
33		1	1		
34				1	
35					1

(4.A)

Prokhorov [Pro05, 1.5] proves that the largest possible genus for a Fano 3-fold with $-K_X$ Cartier is $g = 37$ with degree $-K_X^3 = 2g - 2 = 72$ (strengthening Cheltsov's result $-K_X^3 \le 184$ [Che99]). Moreover Prokhorov proves that $g = 37$ is realised only by $\mathbb{P}(1,1,4,6)$ and $\mathbb{P}(1,1,1,3)$. That completely clears up the final column.

Cheltsov and Karzhemanov extend this to Gorenstein 3-folds in genus $g \ge 33$.

Theorem 4.1 ([Kar09])**.** *Let X be a Fano 3-fold with Gorenstein canonical singularities.*

(1) If $g = 36$ then $X \cong \mathrm{Bl}_p\, \mathbb{P}(1,1,4,6)$, where $p \in \mathbb{P}(4,6)$ is an index 2 point.
(2) The case $g = 35$ is not possible.
(3) If $g = 34$ then X is the anticanonical image of the projectivised bundle $U = \mathrm{Proj}_{\mathbb{P}^1}\, \mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(5)$.

4.2.1. *Genus 34.* We consider Karzhemanov's degree 66 example Z in light of other Fano 3-folds in the GRDB. The weighted projective space $X = \mathbb{P}(1^2, 3, 5)$ is a Fano 3-fold with a terminal singularity $\frac{1}{3}(1,1,2)$, a canonical singularity $\frac{1}{5}(1,1,3)$, genus 34 and degree $-K^3 = 66\frac{2}{3}$. Its Hilbert series lies in $\mathcal{F}_{\mathrm{MF}}$ but not $\mathcal{F}_{\mathrm{ss}}$: it fails the semistability condition, so does not appear in Figure 1, though it is a Fano 3-fold and has anticanonical embedding

$$X \subset \mathbb{P}(1^{35}, 2, 3)$$

in the blank position $(34, 34)$ in (4.A). It fits into a diagram of toric varieties and maps:

$$\begin{array}{ccccccccc}
 & & & & W & & & & \\
 & & & \swarrow & & \searrow & & & \\
U & \overset{\text{flip}}{\dashleftarrow} & Y & & & & V & \overset{\text{flop}}{\dashrightarrow} & V^+ \\
\downarrow & & & \searrow & & \swarrow & & \searrow \quad \swarrow & \\
\mathbb{P}^1 & & & & X & & & Z &
\end{array}$$

where $Y \to X$ is the Kawamata $\frac{1}{3}(1,1,2)$ blowup of $P_3 \in X$ (the index 3 point), $Y \dashrightarrow U$ is the $(5,2,-1,-1)$ (canonical) flip to Karzhemanov's bundle $U \to \mathbb{P}^1$, $V \to X$ is the $\frac{2}{3}$-discrepancy $\frac{1}{3}(2,2,1)$ blowup of $P_3 \in X$ and $V \dashrightarrow V^+$ is the $(5,1,-3,-3)$ (canonical) flop, while finally W is the resolution of $P_3 \in X$, a flop of which admits divisorial contractions to both U and V^+. In this picture

$$\big(X \subset \mathbb{P}(1^{35}, 2, 3)\big) \dashrightarrow \big(Y \subset \mathbb{P}(1^{35}, 2)\big) \dashrightarrow \big(Z \subset \mathbb{P}^{34}\big) \qquad (4.B)$$

is a sequence of projections of Fano 3-folds of degrees $66\frac{2}{3}$, $66\frac{1}{2}$ and 66, where $\rho_X = \rho_Z = 1$ and $\rho_Y = 2$, and X and Y are $\mathbb{Q}$-factorial while Z is not.

Rather loosely speaking, we see how knowing Karzhemanov's example provides other Fano 3-folds by birational contractions, while, from the other end, knowing $\mathbb{P}(1^2, 2, 3)$ provides other Fano 3-folds by birational blow ups; we discuss this further in §4.5.4.

4.2.2. *Genus 33.* Karzhemanov [Kar15] also classifies the case of degree 64. The GRDB matches 5 toric cases, and we illustrate with a beautiful non-$\mathbb{Q}$-factorial example

$$\Phi_{-K_X} : X = \mathrm{TorVar}_{\binom{5}{6}} \begin{pmatrix} 0 & 3 & 5 & 1 & 1 \\ 1 & 4 & 6 & 1 & 0 \end{pmatrix} \subset \mathbb{P}^{34}.$$

(The notation $\mathrm{TorVar}_v\, M_{2\times r}$ denotes the toric variety $\mathbb{C}^r /\!\!/_v (\mathbb{C}^*)^2$, where $(\mathbb{C}^*)^2$ acts by weights that are the rows of M; see [BCZ04, §A].) This is the base of a $(6,1,-5,-2)$ (canonical) flop

$$\begin{array}{ccccccc} & & U & \overset{\text{flop}}{\dashrightarrow} & U^+ & & \\ & \swarrow & & \searrow \quad \swarrow & & \searrow & \\ \mathbb{P}(1^2,4,6) & & & X & & & \mathbb{P}(1^2,3,5) \end{array}$$

where $U \to \mathbb{P}(1^2,4,6)$ is the blowup of a smooth point and $U^+ \to \mathbb{P}(1^2,3,5)$ is a weighted $\frac{1}{3}(1,2,4)$-blowup of the index 3 point; that is,

$$\text{blowup } (-1,-4,-6) \text{ in the cone } \langle (1,0,0),(0,0,1),(-1,-3,-5)\rangle\,.$$

4.3. **Index 2 singularities.** The list $\mathcal{F}_{\mathrm{MF}}$ contains 360 pairs $(g,\mathcal{B})$, where $\mathcal{B} = \{N \times \frac{1}{2}(1,1,1)\}$ with $N \geq 1$ of which 272 lie in $\mathcal{F}_{\mathrm{ss}}$. If we restrict to $g \geq 2$, then these numbers reduce to 325 and 238 respectively.

Sano [San95, San96] and Campana–Flenner [CF93] classify terminal Fano 3-folds X under the assumption $F(X) \geq 1$. For baskets $\{N \times \frac{1}{2}(1,1,1)\}$, these are

$$\begin{aligned} \mathbb{P}(1^2,2^2,3) \supset X_6 &\hookrightarrow \mathbb{P}(1^8,2^3) \quad \text{(anticanonical embedding)} \\ \mathbb{P}(1^3,2^2) \supset X_4 &\hookrightarrow \mathbb{P}(1^{16},2^2) \\ \mathbb{P}(1^4,2) \supset X_3 &\hookrightarrow \mathbb{P}(1^{23},2) \\ \mathbb{P}(1^3,2) &\hookrightarrow \mathbb{P}(1^{34},2) \end{aligned}$$

and when $F(X) = 1$ there are a dozen more subtle $\mathbb{Z}/2$ quotients of smooth Fano 3-folds, also in rather high codimension. (See §4.4 for higher-index more generally.)

The remaining cases for index 2 baskets satisfy $I(X) = 2$ and $F(X) = 1/2$. Takagi [Tak02] classifies Mori–Fano 3-folds with such genus–basket pairs under these conditions with $g \geq 2$. The result is precisely 35 families matching 23 of these elements of $\mathcal{F}_{\mathrm{ss}}$. They are presented in Tables 1–5 of [Tak02], with the individual families numbered 1.1, 1.2,. . . , 5.5. They are listed in Figure 3, ranging from Family 3.1, $X_5 \subset \mathbb{P}(1^4,2)$ to Family 1.14, $X \subset \mathbb{P}(1^{10},2^2)$ in codimension 8, with Type I projections going up the columns.

Codimension \ Genus	2	3	4	5	6	7	8
1	3.1						
2	3.2	5.2					
3	2.1	5.3	4.3				
4	2.2, 3.3	4.1, 5.1	1.3, 4.4	1.4			
5	2.3, 3.4, 5.1	4.2, 5.5	1.2, 1.3, 4.5	1.5, 1.6	1.9, 1.10		
6	2.4		4.6	1.7, 1.8	1.11	1.12	
7			4.7				1.13, 4.8
8							1.14

FIGURE 3. Families of Tables $n.m$ of [Tak02] as they appear in Figure 1, arranged by genus g and codimension c. The generic member of each family is embedded as $X \subset \mathbb{P}(1^{g+2},2^N)$ with basket $\mathcal{B} = \{N \times \frac{1}{2}(1,1,1)\}$ where $N = c - g + 2$.

The comparison with the geography in Figure 1 is striking. The 272 pairs $(g,\mathcal{B}) \in \mathcal{F}_{\mathrm{ss}}$ are spread over most of the table, away from the top diagonal line of Gorenstein pairs $(g,\emptyset)$, and Takagi's result shows that most are not realised by Mori–Fano 3-folds. However, we see in 4.5 that many of the remaining $272 - 23 = 249$ pairs are realised by more general Fano 3-folds, and the Gorenstein index 2 classification remains unknown.

4.4. **Higher Fano index.** Among all Fano 3-folds X, there are some that have divisible canonical class, and we indicate the Hilbert series of those in Figure 4.

There are different possible notions of divisibility. We consider the following: X has divisible anticanonical class if $-K_X = \iota A$ for some ample Weil divisor A and integer $\iota \geq 2$. The graded ring $R(X,A) = \oplus_{m\geq 0} H^0(X,mA)$ is Gorenstein as $H^0(X,-K_X) \subset R(X,A)$ [GW78, 5.1.9]. Suzuki [Suz04, BS07a, BS07b] carries out the analysis to find a set of possible baskets $\mathcal{B}_\iota$ for each ι, with

Codimension \ Genus	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1																																								
2		1	3		1																																			
3								1																																
4			2		1																																			
5			5	2		2																																		
6		5	7	5	1				1																															
7			4			2						1																												
8		6	12	8	2	1	3		2	1																														
9			7		1	4				2																														
10		10	15	13	11		4	1	1				1																											
11		4	14	1	1	4		5		2						1																								
12		6	18	20	12	1	8		1		5		1																											
13		7	14	3	2	15		6		3				3																										
14		10	31	50	23	2	10	1	11	1	3	1	2				2																							
15		11	28	6	3	25	4	12		7	1	5	1	2						1																				
16		12	24	64	37	1	25	2	12		7	1	1		3		1																							
17		18	11	3	4	44	3	16	1	17	2	8	1	4				2																						
18		17	12	44	67	5	42	1	22	3	9	1	8	2	5		2	1	1		1																			
19		13	5		2	40	3	47		21		14		3		5		3						1																
20		2			14	1	38	2	26	1	25	2	16		7	1	2		3	1	1			1																
21						9		16	1	24	3	19	2	19		6		4				2																		
22									8		14	3	13	1	12		8		4	1	2				2															
23										1		7	1	9	1	11	1	4		8		2		1				1												
24													1		6		8	1	7	1	4		3		3															
25																		8	1	5	1	4				2														
26																			1		4		4		1		1		1		1									
27																						1	1	3	2	3						1								
28																									1		3		2											
29																														2										
30																													1			1	1							
31																																			1	2				
32																																		1						
33																																								
34																																								
35																																								1

FIGURE 4. Number of Hilbert series of semistable Fano 3-folds of index ≥ 2 listed by genus and estimated minimal codimension.

additional genus information when $\iota \leq 2$. Again there is a semistability condition that can be imposed, which we do here. The numbers of baskets (or basket–genus pairs for $\iota \leq 2$) per index $\iota \geq 1$ is:

ι	1	2	3	4	5	6	7	8	9	11	13	17	19
$\#\mathcal{B}_\iota$	39550	1413	181	82	34	6	12	4	2	3	2	1	1

There are no Fano 3-folds of indices $\iota = 12$, 14, 15, 16 or 18 by [Suz04], and also no semistable case – in fact, [Pro10] proves that there is no Mori–Fano 3-fold of index 10, semistable or not, but that is much more sophisticated information.

The point here is that if $-K_X = \iota A$, then the A-embedded model $\operatorname{Proj} R(X, A)$ is usually much simpler that the anticanonical ring. The numbers in low codimension are:

ι_X	2	3	4	5	6	7	8	9	11	13	17	19	total
codim 0	0	0	1	1	0	1	0	0	1	1	1	1	7
codim 1	8	7	2	5	1	4	3	2	2	1			35
codim 2	26	6	7	1									40

(4.C)

Example 4.2. The codimension 0 row of (4.C) are the 7 weighted projective spaces

	$\mathbb{P}^3$	$\mathbb{P}(1^3,2)$	$\mathbb{P}(1^2,2,3)$	$\mathbb{P}(1,2,3,5)$	$\mathbb{P}(1,3,4,5)$	$\mathbb{P}(2,3,5,7)$	$\mathbb{P}(3,4,5,7)$
$-K^3$	64	125/2	343/6	1331/30	2197/60	4913/210	6859/420
g	33	32	29	22	18	11	7
cod	31	31	30	27	25	22	20
W	1^{35}	$1^{34},2$	$1^{31},2^2,3$	$1^{24},2^4,3^2,5$	$1^{20},2^4,3^3,4,5$	$1^{13},2^5,3^4,4^2,5,7$	$1^9,2^7,3^4,4,5^2,7$

and these embed anticanonically in $\mathbb{P}(W)$ as the model given in GRDB.

Example 4.3. The A-hypersurfaces usually embed in a bigger space under their anticanonical ι-Veronese embedding than GRDB suggests. For example, $X_{10} \subset \mathbb{P}(1^2,2,3,5)$ has $\iota_X = 2$, so $-K_X = 2A$ and the anticanonical embedding is

$$X \xrightarrow{\cong} \Phi_{2A}(X) \subset \mathbb{P}(1^4,2^2,3^3,4)$$

in codimension 6. Its ($-K_X$-polarised) Hilbert series, however, admits a simpler model $Y_{4,4} \subset \mathbb{P}(1^4,2,3)$, and this is the model one given in GRDB.

In high codimension, Gorenstein projection does not work in the same way as the $\iota = 1$ case (since the image of projection has non-isolated singularities), so the GRDB suggests models in the total A-embedding by comparing with the K3 database [Bro07]: $S \in |\iota A|$ has the numerical properties of a K3 surface polarised by $A_{|S}$, so a choice of weights is determined by [Bro07]; then including a variable of degree ι (noting that it may then be eliminated by an equation of degree ι) gives a choice of weights for X.

4.5. **Toric Fano 3-folds.** Kasprzyk [Kas10a] classifies toric Fano 3-folds with canonical singularities as a list $\mathcal{T}_{\text{can}}$ of 674,688 lattice polytopes; this classification is available from [Kas10b]. These varieties are all $\mathbb{Q}$-Gorenstein, though only 12,190 are $\mathbb{Q}$-factorial.

The GRDB contains $\mathcal{T}_{\text{can}}$, and it can be analysed online [BK09]. More importantly, the GRDB connects $\mathcal{T}_{\text{can}}$ with the Fano 3-fold database: each polytope $\Delta \in \mathcal{T}_{\text{can}}$ has a Hilbert series $P_\Delta \in \mathcal{F}_{\text{MF}}$, all but 12 of which lie in $\mathcal{F}_{\text{ss}}$. These 12 all have isolated, $\mathbb{Q}$-factorial, strictly canonical singularities, and are either one of the 6 weighted projective spaces

$$\mathbb{P}(1,1,3,5),\ \mathbb{P}(1,2,5,7),\ \mathbb{P}(1,3,7,10),\ \mathbb{P}(1,3,7,11),\ \mathbb{P}(1,5,7,13),\ \mathbb{P}(3,5,11,19)$$

or a rank 2 blowup of one of these. The GRDB links Δ to P_Δ, and conversely for any $P \in \mathcal{F}_{\text{ss}}$ reports all those Δ with $P_\Delta = P$. This matching is significant in different ways, as we discuss next.

4.5.1. *Location in Geography.* The toric Fano 3-folds in $\mathcal{T}_{\text{can}}$ populate large areas of Figure 1 with Fano 3-folds. We draw the submap of Hilbert series that are realised by at least one toric Fano 3-fold in Figure 6.

Only 8 of the toric cases are Mori–Fano 3-folds: the 7 weighted projective spaces of 4.2 together with the magical fake weighted projective space of degree 64/5 and genus 5

$$\mathbb{P}^3/\mathbb{Z}/5(1,2,3,4) \hookrightarrow \mathbb{P}(1^7,2^8,3^4,5^4)$$

in codimension 19. Including these, there are 634 terminal cases and 233 terminal $\mathbb{Q}$-factorial cases of higher Picard rank; see Figure 5 for the high-codimension cases.

For some Hilbert series there are many matching toric 3-folds, and this is recorded on the GRDB, with an idea of the multiplicities in Figure 7. It seems amazing to us that two different polytopes can contain the same number of lattice points at all dilations – but of course whenever two polytopes are mutation equivalent [ACGK12], exactly this happens.

There are 4319 varieties $X \in \mathcal{T}_{\text{can}}$ that have $-K_X$ Cartier (cf. [KS98]); of these, 194 are $\mathbb{Q}$-factorial. The number of these is listed by genus in the column $\#\mathcal{T}$ of Figure 2. We have not checked whether any satisfy Petracci's non-smoothability condition [Pet20, 1.1], nor whether any lie at the intersection of multiple smooth families (see §4.5.3).

4.5.2. *Toric degenerations.* Some approaches to or applications of Mirror Symmetry require toric degenerations of Fano 3-folds [CI16, CCGK16]. The link between lists summarised in Figure 6 is a necessary condition for a Fano 3-fold X with given Hilbert series to have a toric Fano 3-fold degeneration X_0. This numerical condition is not sufficient, as it does not determine whether X and X_0 lie in the same deformation family, for example.

It would be natural to extend this correspondence either to include more general toric varieties, or reducible varieties composed of toric varieties glued along toric strata.

4.5.3. *Deformation of toric varieties and intersections of families.* Following Altmann's local analysis [Alt97, Alt00], a lot is known about how toric varieties deform. For toric Fano 3-folds with isolated singularities, global deformations surject onto local deformations [Pet19, 2.3], so understanding the deformation theory of singularities on toric Fano 3-folds is a powerful tool.

Example 4.4. The first element $X_1 \in \mathcal{T}_{\text{can}}$ has the Hilbert series of some $X \subset \mathbb{P}(1^7, 2^4)$ with a basket $4 \times \frac{1}{2}(1,1,1)$. However, X_1 is not quasismooth: it has six Gorenstein facets that are the cone on the del Pezzo surface of degree 6 and four cones of type $\frac{1}{2}(1,1,1)$. Each of these del Pezzo cone singularities has two smoothing components locally, so since deformations of toric Fano 3-folds surject onto local deformations, there are at least seven distinct quasi-smoothing components that contain different small deformations of X_1; compare 4.6 below.

Extending this to Gorenstein index 2 toric Fano 3-folds that also have isolated cone over del Pezzo degree 6 singularities gives three more examples with two distinct quasi-smoothing families:

$\mathcal{T}_{\text{can}}$(id)	$\#\frac{1}{2}$	$\#\text{dP}_6$	$\mathcal{F}_{\text{ss}}$(id)	$X \subset w\mathbb{P}$	g	codim
1	4	6	27334	$\mathbb{P}(1^7, 2^4)$	5	7
254482	3	1	38250	$\mathbb{P}(1^{17}, 2^3)$	15	16
254485	6	1	36639	$\mathbb{P}(1^{13}, 2^6)$	11	15
254810	3	1	38935	$\mathbb{P}(1^{20}, 2^3)$	18	19

4.5.4. *High codimension representatives and cascades.* Many of the weights in GRDB are constructed inductively by considering a single projection, but varieties frequently arise in sequences, or *cascades*, of projections: famously $X = \Phi_{-K}(\mathbb{P}^2) \subset \mathbb{P}^9$ has sequences of projections from points (which in this case are blowups) that recover elements of most families of del Pezzo surfaces; see [RS03] for extensions. It seems typical that the ends of such cascades are simpler to describe than the middles: things like toric varieties live at the top, while hypersurfaces live at the bottom. Thus any $X \in \mathcal{T}_{\text{can}}$ of high codimension for its genus may be a good candidate for the head of a cascade.

For example, in $g = 8$, the highest codimension $\mathbb{Q}$-factorial terminal Fano 3-fold is

$$\mathcal{T}_{\text{can}}(544385)\colon \mathbb{P}^1 \times \mathbb{P}^2/\tfrac{1}{3}(0,1,0,1,2) \subset \mathbb{P}(1^{10}, 2^6, 3^6)$$

of Picard rank 2 in codimension 18, with $6 \times \frac{1}{3}(1,1,2)$ singularities at the 6 toric 0-strata. The Fano polytope is the simplicial decomposition on vertices

$$(1,0,0), (0,1,0), (-1,-1,0), (1,2,3), (-1,-2,-3)$$

with six index 3 cones meeting at a central '$\mathbb{P}^2$' triangular equator with a cycle of three northern cones and three southern cones with Sym_3 symmetry. The Kawamata blowup of any one of the index 3 points is equivalent to any other, and gives the first projection. There are four ways to project from a pair of index 3 points, depending on adjacency; in the case of the blowup of a northern cone and the adjacent southern cone, the equator becomes a flopping curve that is contracted to an ordinary node. Continuing, the projection from all index 3 points gives a variety

$$Y \subset \mathbb{P}(1^{10}, 2^6)$$

in codimension 12 with $6 \times \frac{1}{2}(1,1,1)$ singularities and 3 nodes. This variety admits a quasi-smoothing, so is a Gorenstein index 2 Fano 3-fold that does not appear in Figure 3, as it has $\rho_Y > 1$. Further projections from index 2 points give more Gorenstein index 2 varieties that extend Figure 3 in genus 8 to Fano 3-folds that are not Mori–Fano.

The highest-codimension toric Fano 3-folds with terminal singularities by genus – that is, the toric candidates for the top of terminal cascades – are listed in Figure 5.

$\mathcal{T}_{\mathrm{can}}$(id)	g	ρ_X	$\mathcal{F}_{\mathrm{ss}}$(id)	$X \subset w\mathbb{P}$	$\mathcal{B}$	codim	
547383	5	1	29211	$\mathbb{P}(1^7, 2^8, 3^4, 5^4)$	$4 \times \frac{2}{5}$	19	$\mathbb{P}^3/\frac{1}{5}(1,2,3,4)$
547379	7	1	32734	$\mathbb{P}(1^9, 2^7, 3^4, 4, 5^2, 7)$	$\frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{3}{7}$	20	$\mathbb{P}(3,4,5,7)$
544385	8	2	33967	$\mathbb{P}(1^{10}, 2^6, 3^6)$	$6 \times \frac{1}{3}$	18	
547380	11	1	36623	$\mathbb{P}(1^{13}, 2^5, 3^4, 4^2, 5, 7)$	$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{3}{7}$	22	$\mathbb{P}(2,3,5,7)$
430483	12	4	36948	$\mathbb{P}(1^{14}, 2^3, 3^3)$	$3 \times \frac{1}{3}$	16	
520102	13	3	37585	$\mathbb{P}(1^{15}, 2^4, 3^3, 4)$	$\frac{1}{2}, 2 \times \frac{1}{3}, \frac{1}{4}$	19	
544370	14	2	38020	$\mathbb{P}(1^{16}, 2^4, 3^3, 4, 5)$	$\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$	21	
544376	15	2	38404	$\mathbb{P}(1^{17}, 2^5, 3^2, 5)$	$2 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{5}$	21	
430473	16	4	38533	$\mathbb{P}(1^{18}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	18	
520107	17	3	38760	$\mathbb{P}(1^{19}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	19	
520148	17	2	38760	as previous			
547382	18	1	39006	$\mathbb{P}(1^{20}, 2^4, 3^3, 4, 5)$	$\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$	25	$\mathbb{P}(1,3,4,5)$
520124	19	3	39052	$\mathbb{P}(1^{21}, 2^2, 3)$	$\frac{1}{2}, \frac{1}{3}$	20	
520103	20	3	39192	$\mathbb{P}(1^{22}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	22	
544394	21	1	39278	$\mathbb{P}(1^{23}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	23	$X_4 \subset \mathbb{P}(1^2, 2^2, 3)$
547381	22	1	39368	$\mathbb{P}(1^{24}, 2^4, 3^2, 5)$	$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}$	27	$\mathbb{P}(1,2,3,5)$
520128	24	3	39416	$\mathbb{P}(1^{26}, 2)$	$\frac{1}{2}$	23	
520131	24	3	39416	as previous			
544383	25	2	39457	$\mathbb{P}(1^{27}, 2^2, 3)$	$\frac{1}{2}, \frac{1}{3}$	26	
544389	26	2	39476	$\mathbb{P}(1^{28}, 2)$	$\frac{1}{2}$	25	
544388	28	2	39510	$\mathbb{P}(1^{30}, 2)$	$\frac{1}{2}$	27	
547384	29	1	39526	$\mathbb{P}(1^{31}, 2^2, 3)$	$\frac{1}{2}, \frac{1}{3}$	30	$\mathbb{P}(1^2, 2, 3)$
547385	32	1	39541	$\mathbb{P}(1^{34}, 2)$	$\frac{1}{2}$	31	$\mathbb{P}(1^3, 2)$

FIGURE 5. High-codimension non-Gorenstein $\mathbb{Q}$-factorial terminal toric Fano 3-folds.

Beyond toric, [BHHN16, BHHN17] initiates the analysis of low complexity Fano varieties, with the classification of Picard rank 1, $\mathbb{Q}$-factorial terminal Fano 3-folds of complexity 1. As in 4.3, these are almost all hypersurfaces with high-codimension anticanonical embedding, where the complexity condition enforces very particular trinomial equations.

4.6. **Formats and low codimension.** All $95 + 85$ Hilbert series in $\mathcal{F}_{\mathrm{MF}}$ whose GRDB model is in codimension 1 or 2 actually lie in $\mathcal{F}_{\mathrm{ss}}$ and may be constructed by hand as complete intersections as proposed; these are the first two rows of Figure 1. These varieties all have Picard rank 1 by the Lefschetz hyperplane theorem (compare [CPR00, 3.5]).

The same is true of all 70 Hilbert series with codimension 3 models, which occupy the third row of Figure 1. In that case, only $X_{2,2,2} \subset \mathbb{P}^6$ is a complete intersection. The remaining 69 cases are cut out by the five maximal Pfaffians of a skew 5×5 matrix. Corti and Reid [CR02], following Grojnowski, explain this as a pullback from a weighted Grassmannian $w\mathrm{Grass}(2,5)$ in a precise sense, which informally we may treat as saying that the Plücker embedding $\mathrm{Grass}(2,5) \subset \mathbb{P}^9$ is described by the Pfaffians of a generic skew 5×5 matrix of linear forms, and we may specialise these forms as we please, taking care with homogeneity. Again, these varieties have Picard rank 1 by [BF20, 3].

This idea leads to the general idea of 'format' [BKZ19], where the equations (and syzygies, and indeed the whole minimal free resolution) of a 'key variety' (that is, any variety you like) are used as a model for the equations of other varieties by graded pullback.

This idea is implemented in several places; [CR02, QS11, BKZ19, CD20], for example. One point that arises is that the Picard rank should be inherited from the format, and so it is possible to target Fano 3-folds of different rank.

Example 4.5. In [BKQ18, 1.2], the variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ in its Segre embedding is used as a key variety to model some Fano 3-folds in codimension 4 that have Picard rank 2. That analysis constructs examples of deformation families in different codimension for the same Hilbert series. For example, $\mathcal{F}_{\mathrm{ss}}(548)$ is presented in GRDB as

$$X \subset \mathbb{P}(1,3,4,5,6,7,10)$$

a codimension 3 Pfaffian that is easy to construct, but there is another family

$$X' \subset \mathbb{P}(1,3,4,5,6,7,9,10)$$

which arises from unprojection of a degeneration. One may suspect that such Fano 3-folds are degenerations of the Pfaffian model, but this is not the case: quasismooth members of the two families have different invariants, such as Picard rank $h^{1,1}(X)$ and Euler characteristic e_X, so cannot lie in a common flat family.

Different formats cover everything in codimension ≤ 3 much in codimension 4 (see [CD20, 5.3], where a cluster variety format describes certain subfamilies of deformations), and some in codimension 5. but although there are examples in higher codimension, they seem to realise only a small part of the classification there – see [QS11, 4.4, 5.2], where flag variety formats recover classical smooth Fano 3-folds but no other element of $\mathcal{F}_{\rm ss}$ has suitable Hilbert series, or [CD20, 5.8], where a codimension 6 format realises no Fano 3-folds. (Of course, such failures could be because there are few Fano 3-folds in high codimension – we simply do not know.)

4.7. **Implementing unprojection.** The unprojection ansatz in §3.2.1 can sometimes be realised: given a coordinate plane $D = \mathbb{P}(a_i, a_j, a_k) \subset \mathbb{P}(a_0, \dots, a_n)$, one may be able to construct a Fano 3-fold X that contains D with X quasismooth away from finitely many nodes on D. In this model case, the Type I unprojection constructs a quasismooth Fano 3-fold $Y \subset \mathbb{P}(a_0, \dots, a_n, r)$ (the mild numerical conditions of §3.2 are only to exclude quasilinear equations in the ideal of Y).

This has been carried out systematically in [BKR12] in the case $X \subset w\mathbb{P}^6$ lies in codimension 3, so that $Y \subset w\mathbb{P}^7$ lies in codimension 4, with further cases in [BD]. Type II unprojections are considered in [Rei02, Pap08, Tay], which construct other cases in codimension 4. Using this and §4.6, for every Hilbert series of $\mathcal{F}_{\rm ss}$ whose GRDB model is in codimension 4, there is a construction of a variety that matches that model, and in the majority of cases there are 2 or more distinct deformation families.

4.8. **The Fanosearch programme.** Coates, Corti, Galkin, Golyshev, Kasprzyk [CCG+13] and others, following ideas of Golyshev [Gol07], provide an alternative approach to the Fano classification problem. The idea is that via Mirror Symmetry, Fano classification can be rephrased as a fundamentally combinatorial problem of identifying suitable Laurent polynomials whose periods generate solutions to Picard–Fuchs equations on the other side of the mirror. Although the combinatorial problems seem hard, and the required mirror theorems are not wholly in place, [CCGK16] confirms that the two sides of the mirror agree in the smooth case, and in doing so provides a wealth of tools for constructing Fano varieties and passing through the mirror. The GRDB is a key tool: see e.g. [CHKP, Heu].

Example 4.6. Let P be the Fano polytope whose spanning fan gives rise to $X_1 \in \mathcal{T}_{\rm can}$. Using the terminology of [CKPT21], P supports eleven rigid maximally mutable Laurent polynomials (rigid MMLPs), up to automorphisms of P. The periods of these rigid MMLPs give solutions to eleven distinct Picard–Fuchs equations; compare this with Example 4.4, where prima facie we see seven deformation families. By [CKPT21, Conjecture 5.1] we expect each of these eleven rigid MMLPs to correspond to a deformation of X_1 to a terminal locally toric Fano with $4 \times \frac{1}{2}(1,1,1)$ singularities. This expectation agrees with the output of Ilten's Macaulay2 package [Ilt12]. More generally, considering those $X \in \mathcal{T}_{\rm can}$ with Hilbert series equal to that of X_1 gives a total of 24 rigid MMLPs, up to mutation, corresponding to 24 distinct Picard–Fuchs equations and hence, conjecturally, at least 24 deformation families of terminal Fano 3-folds with basket $4 \times \frac{1}{2}(1,1,1)$.

5. Synopsis

5.1. **Review of guiding examples.** The Fano 3-fold database, the two lists of Hilbert series $\mathcal{F}_{\rm ss} \subset \mathcal{F}_{\rm MF}$ together with the estimated weights $X \subset \mathbb{P}(a_0, \dots, a_n)$ that the GRDB assigns to each one, is intended as a first coarse approximation to the classification of Mori–Fano 3-folds. However, it is certainly nowhere near to a final classification. The following remarks and examples are intended as quick reminders to help avoid misunderstandings.

(1) Overview of the Fano 3-fold database:
   (a) We distinguish between Mori–Fano 3-folds (outcomes of the Minimal Model Program; Definition 1.1) and Fano 3-folds more generally (Definition 2.1).
   (b) The Fano 3-fold database is a set $\mathcal{F}_{\rm ss}$ of rational functions that satisfy the numerical conditions of [Kaw92] that constrain the Hilbert series of semistable Mori–Fano 3-folds (§2). A larger set $\mathcal{F}_{\rm MF} \supset \mathcal{F}_{\rm ss}$ allows for some strictly non-semistable cases. The geography of Figure 1 is of $\mathcal{F}_{\rm ss}$ only. We do not know an example of a Mori–Fano 3-fold not in $\mathcal{F}_{\rm ss}$.

(c) Although the main consideration is Mori–Fano 3-folds, we are interested in recording any Fano 3-folds that realise elements of $\mathcal{F}_{\mathrm{MF}}$.
(d) A series $P \in \mathcal{F}_{\mathrm{ss}}$ may be realised by many deformation families (Figures 2, 3) or by none (4.1(2)): $\mathcal{F}_{\mathrm{ss}}$ does not count the number of deformation families. This is a basic part of the classification problem, and it is fully understood only in the case of nonsingular Fano 3-folds and some specific cases with only $\frac{1}{2}(1,1,1)$ singularities.

(2) Existence and non existence:
(a) Proven cases of Mori–Fano 3-folds are sparse: the smooth Fano 3-folds of Picard rank 1 (§4.1), the Gorenstein index 2 classification (§4.3), and a range of cases in low anticanonical codimension (§4.6) or of high Fano index (§4.4). These reveal many locations in Figure 1 that are not realised by a Mori–Fano 3-fold.
(b) There is no reason why the Hilbert series of more general Fano 3-folds should appear in $\mathcal{F}_{\mathrm{ss}}$ or $\mathcal{F}_{\mathrm{MF}}$, though this is the case for every example we know.
(c) More general Fano 3-folds provide many more examples throughout $\mathcal{F}_{\mathrm{ss}}$: many locations in Figure 1 are realised by a Fano 3-fold but not by a Mori–Fano 3-fold.
(d) We expect that many of the high codimension, lower genus Hilbert series are not realised even by a Fano 3-fold. However, we only know one place in Figure 1 where this is proven: Karzhemanov's nonexistence result for genus 35 (§4.2).

(3) The estimated anticanonical embedding $X \subset \mathbb{P}(a_0, \ldots, a_n)$:
(a) The Hilbert series P_X of a Fano 3-fold X does not determine the weights $a_0, \ldots, a_n$ of its anticanonical embedding (4.5).
(b) The embedding weights given to each $P \in \mathcal{F}_{\mathrm{MF}}$ in the GRDB are only a suggestion. They are derived from known examples in low codimension (§3.1.1), an analysis of conjectured Gorenstein projections (§3.2) and, in harder cases, an analysis of singularities or the linear systems on possible K3 sections. Although the weights are often right, there is no reason why your X should be embedded in this way.
(c) Even if the general member of a deformation family is $X \subset \mathbb{P}(a_0, \ldots, a_n)$, there are likely to be degenerations in higher codimension (§3.1.1).
(d) It can happen that $P \in \mathcal{F}_{\mathrm{ss}}$ has distinct deformation families whose general members embed in different codimensions (4.5).
(e) Some higher index Fano 3-folds lie in higher codimension than GRDB suggests (4.3).

5.2. **Nonexistence and other challenges.** The GRDB is simply one way of assembling and presenting the vast amount of data associated to the classification of Fano 3-folds, and as such it naturally invites more questions than it answers. A selection of topics:

(1) Find $P \in \mathcal{F}_{\mathrm{ss}}$ not realised by a Fano 3-fold, or not realised by a Mori–Fano 3-fold.
(2) Can one show that each model for $P \in \mathcal{F}_{\mathrm{ss}}$ in codimension 5 or 6 is realised by a Fano 3-fold using similar birational methods as in codimension 4?
(3) There are toric Fano 3-folds in high codimension: can projection from these realise Fano 3-folds in sequences of Hilbert series?
(4) Fano 3-folds with $|-K_X|$ empty are rare. The GRDB has 264 semistable Hilbert series with linear coefficient zero (the left-hand column $g = -2$ of Figure 1). The first are Iano–Fletcher's example $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ and the three families in codimension 4, studied by [AR00, Pap08, Tay], though they remain to be fully understood. The next model with no Type II projection attack is $X \subset \mathbb{P}(2,3,4^2,5^2,6^2,7^2)$ in codimension 6.
(5) The GRDB is constructed with Gorenstein projection in mind, and the Fano 3-fold database includes that data; click [BK09]. Sarkisov links provide another connection between Fano 3-folds, and projection is often the first step of a Sarkisov link. Classification attacks often exploit such links, [Tak89] for example, but does it even make sense to describe a web of Sarkisov links overlaying the GRDB?
(6) The bounds of [Kaw92] work over any field k of characteristic 0, not necessarily algebraically closed. Thus, for example, the GRDB makes sense over $k = \mathbb{Q}$ (though it is not clear that the more complicated unprojection constructions are also defined over $\mathbb{Q}$), and the generic fibres of relatively 3-dimensional Mori fibre spaces also have relative Hilbert series in the Fano 3-fold database.

5.3. **Closing repetition of the main warning.** It is easy to mistake the Fano 3-fold database for a classification of Mori–Fano 3-folds. It is **not** that classification: that classification does not yet exist.

It is instead the classification of genus–basket pairs that satisfy certain conditions of geometric origin, or equivalently it is a list of the rational functions they determine by the plurigenus formula.

The confusion arises in part because each rational function is presented as though it is the Hilbert series of a Fano 3-fold $X \subset \mathbb{P}(a_0, \dots, a_n)$ embedded by its total anticanonical ring in weighted projective space with given weights. This description is sometimes accurate, but in fact the weights are simply a convenient informed estimate. There is no claim that a Fano 3-fold exists with this data, nor that a particular one you may be considering is necessarily embedded anticanonically as indicated here. Furthermore, a single Hilbert series may be realised by more than one family of varieties, and these multitudes may lie in different ambient weighted projective spaces.

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Codimension \ Genus	−2 ⋯ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
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12	1	45	65	47	35	21	15	8	6	3	2	1	1																							
13		59	86	71	48	35	21	15	8	6	3	2	1	1																						
14		54	126	92	72	48	35	21	15	8	6	3	2	1	1																					
15		51	148	120	89	74	48	35	21	15	8	6	3	2	1	1																				
16		33	161	127	107	93	72	46	35	21	15	8	6	3	2	1	1																			
17		15	100	102	84	128	87	69	46	35	21	15	8	6	3	2	1	1																		
18		3	33	37	32	131	112	84	66	42	33	21	15	8	6	3	2	1	1																	
19		1	4	10	2	122	88	95	61	54	33	33	20	15	8	6	3	2	1	1																
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FIGURE 6. Number of series in $\mathcal{F}_{ss}$ realised by canonical toric 3-folds (5610 total), listed by genus and codimension.

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Codimension	Genus: −2 ··· 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
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2		1	7																																	
3		3	10	23																																
4		8	55	50	54																															
5		32	162	307	182	135																														
6		76	584	921	927	422	207																													
7		231	1468	3083	2705	1980	745	314																												
8		490	3598	6537	7282	4971	2954	1009	373																											
9		1025	6084	12835	13094	10904	6230	3372	1121	416																										
10		1493	9308	17594	20444	15942	11726	6365	3366	1142	413																									
11		1897	10196	21656	22519	19992	14512	10109	5439	3068	1046	413																								
12	7	1739	9926	17859	21825	18423	15498	10568	7540	4321	2498	904	348																							
13		1302	7133	13379	15659	14817	12117	10204	7041	5492	3292	2017	744	334																						
14		637	4727	7202	9911	9294	8810	7628	5982	4467	3816	2417	1529	622	274																					
15		262	2317	3599	4356	5580	5063	5694	3699	3602	2912	2535	1652	1272	486	234																				
16		72	989	1210	1600	2818	3139	3581	2291	2133	2188	1885	1672	1398	950	386	179																			
17		21	253	389	369	1763	1656	2235	1155	1335	1131	1274	1080	1491	953	657	274	151																		
18		3	51	80	64	751	895	1022	518	532	517	644	601	938	986	668	455	186	117																	
19		1	4	16	2	276	302	390	164	194	135	254	241	621	690	645	412	360	147	87																
20			1	1		68	81	98	36	47	33	53	78	285	557	441	364	300	225	91	66															
21						13	10	21	2	11	3	10	12	106	264	252	187	323	168	120	72	40														
22						1		2		1			1	20	119	85	96	199	185	103	110	32	42													
23														3	32	28	42	182	103	72	80	61	27	27												
24															9	1	13	81	68	29	68	29	27	11	18											
25															1		3	34	17	11	30	26	19	13	16	8										
26																		5	6	1	25	15	18	3	15	5	13									
27																			1		17	10	10	1	5	4	3	9								
28																					2	2	3		1	2	3	2	4							
29																												2	4	2						
30																												1	2	1	2					
31																														1	1	5				
32																														1			1			
33																																	1			
34																																				
35																																				2

Figure 7. Number of canonical toric 3-folds that satisfy the semistable numerical condition, listed by genus and codimension.