THE COMBINATORIAL PICARD GROUP

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These notes are adapted from (Ewald 1996). For more information see also (Fulton 1993, Oda 1978).

Given a fan $\Delta$ of cones in $N_\mathbb{R}$ we shall denote its set of dual cones by $\Delta^\vee := \{ \sigma^\vee \mid \sigma \in \Delta \}$.

To each cone $\sigma \in \Delta$ we assign the commutative semigroup $S_\sigma := \sigma^\vee \cap M$, where $M := \text{Hom} (N, \mathbb{Z})$ is the dual lattice of $N$. We thus obtain a system $S(\Delta) := \{ S_\sigma \mid \sigma \in \Delta \}$ of semigroups assigned to $\Delta$. There exist bijective relations $\Delta \leftrightarrow \Delta^\vee \leftrightarrow S(\Delta)$.

**Lemma 1.** Any $n$-dimensional cone $\sigma$ is the vector sum of a cone $\sigma_0$ with apex $0$ and a linear space $U$, i.e. $\sigma = \sigma_0 + U$, where $\dim \sigma_0 + \dim U = n$. If in addition $\sigma$ is a lattice cone, then $\sigma_0$ and $U$ can also be chosen to be lattice cones.

Such a $U$ is uniquely determined, however for $\dim U > 0$ we can choose $\sigma_0$ in many ways. It suffices to take $U = \sigma \cap (-\sigma)$. We call $U$ the cospan of $\sigma$ and write:

$$U = \text{cospan} \sigma.$$ 

To each $\sigma^\vee \in \Delta^\vee$ we assign $m_\sigma \in M$ such that

$$\text{(I)}$$

if $\tau$ is a face of $\sigma$ then $m_\sigma - m_\tau \in \text{cospan} \tau^\vee$.

We can now replace each sub-semigroup $S_\sigma$ in $S(\Delta)$ by its residue class $m_\sigma + S_\sigma$, preserving the inclusion of semigroups in $S(\Delta)$.

**Definition 1.** A system $P := \{ m_\sigma + \sigma^\vee \}_{\sigma \in \Delta}$ of translated cones, such that (I) is satisfied, is called a virtual polytope (with respect to the fan $\Delta$).

If $\Delta$ is polytopal we may choose $\{ m_\sigma \}_{\sigma \in \Delta}$ such that

$$\bigcap_{\sigma \in \Delta} (m_\sigma + \sigma^\vee) =: P$$

is a lattice polytope, and $-P^*$ spans $\Delta$. In such a case, $P$ and $P$ can be identified. The proofs of the following two lemmas are elementary.

**Lemma 2.** The virtual polytopes with respect to the fan $\Delta$ are a commutative group $\tilde{G}$ with respect to addition defined by

$$P + P' := \{ m_\sigma + m_\sigma' + \sigma^\vee \}_{\sigma \in \Delta}.$$ 

The zero element is $\Delta^\vee$.

**Lemma 3.** The system $M := \{ m_\sigma + S_\sigma \}_{\sigma \in \Delta}$ of residue classes assigned to the semigroups of $S(\Delta)$ define a commutative group $G$ with respect to addition given by

$$M + M' := \{ m_\sigma + m_\sigma' + S_\sigma \}_{\sigma \in \Delta}.$$ 

The zero element is $S(\Delta)$.

The groups $G$ and $\tilde{G}$ are isomorphic.
Many properties of virtual polytopes remain true under translation (applied to all \(m_\sigma + \sigma^\vee, \sigma \in \Delta\) simultaneously). Thus we make the following definition:

**Definition 2.** If \(G\) is the group above, we call \(G/\mathbb{Z}^n\) the combinatorial Picard group, \(\text{Pic} \Delta\) of \(\Delta\). We denote its elements by \(P\).

The Picard group \(\text{Pic} \Delta\) is a finitely generated commutative group, and hence (by the fundamental theorem on commutative groups) is equivalent to the direct sum

\[
\text{Pic} \Delta \cong \mathbb{Z}^g \oplus \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_p},
\]

where \(\mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_p}\) is the torsion of the group, and \(q\) is its Betti number.

**Lemma 4.** For \(\text{Pic} \Delta\) to be a torsion-free group, it is sufficient that \(\Delta\) contains an \(n\)-cone \(\tau\).

**Proof.** Suppose \(\text{Pic} \Delta\) contains an element of finite order. Then there is a virtual polytope \(P = \{m_\sigma + \sigma^\vee\}_{\sigma \in \Delta}\) and a natural number \(r\) such that \(rP = \{rm_\sigma + \sigma^\vee\}_{\sigma \in \Delta}\) can be obtained from \(\Delta^\vee\) by adding a lattice vector \(c\). Since \(\tau\) is \(n\)-dimensional, \(\{rm_\tau + \sigma^\vee\}\) is a lattice point, and hence \(rm_\tau = c\). Since \(m_\tau\) also is a lattice point \(c_0\), \(P = c_0 + \Delta^\vee\), so that \(P\) represents the zero element of \(\text{Pic} \Delta\).

If \(\Delta\) contains an \(n\)-cone we can calculate \(\text{Pic} \Delta\) explicitly. The following two results are taken from (Ewald 1996, pp.171-3).

**Theorem 1.** Let \(\Delta\) be a simplicial fan in \(N_\mathbb{R} \cong \mathbb{R}^n\) which contains at least one \(n\)-cone, and let \(k\) be the number of rays of \(\Delta\). Then

\[
\text{Pic} \Delta \cong \mathbb{Z}^{k-n}.
\]

**Theorem 2.** Let \(\Delta\) be a fan in \(N_\mathbb{R} \cong \mathbb{R}^n\) which contains at least one \(n\)-cone, and let \(\{c_1, \ldots, c_q\}\) be the set of rays of \(\Delta\). We consider all maximal faces \(\{\sigma_1, \ldots, \sigma_q\}\) of \(\Delta\) which are not simplex cones, and set, for \(\sigma_i = \rho_{i_1} + \ldots + \rho_{i_k}, i = 1, \ldots, q\),

\[
L_{\sigma_i} := L(d_{i_1}, \ldots, d_{i_k}) \quad \text{(space of linear dependencies)}
\]

and

\[
L := L_{\sigma_1} + \ldots + L_{\sigma_q}, \quad \lambda := \dim L.
\]

Then

\[
\text{Pic} \Delta \cong \mathbb{Z}^{k-n-\lambda}.
\]

**Definition 3.** We call \(\mu(\Delta) := k - n - \lambda\) the combinatorial Picard number of \(\Delta\).

Let \(\Delta\) be a complete fan. We may choose \(m_\tau = m_\sigma\) if \(\tau\) is a face of \(\sigma \in \Delta^{(n)}\), where, of course, condition (I) must be observed. Hence if \(\Delta^{(n)} = \{\sigma_1, \ldots, \sigma_q\}\) and \(a_i := m_{\sigma_i}, i = 1, \ldots, q\), the cones

\[
\{a_1 + \sigma_1^\vee, \ldots, a_q + \sigma_q^\vee\}
\]

determine an element \(P \in \text{Pic} \Delta\), written

\[
P := [a_1 + \sigma_1^\vee, \ldots, a_q + \sigma_q^\vee].
\]

**Definition 4.** We call \(P\) an associated polytope of \(\Delta\) if \(\Delta = \Delta(-P)\). That is, if \(\Delta\) is spanned by \(-P^*\), or in other words, if \(\Delta\) is the fan of normal cones of \(-P\).

**Lemma 5.** Let \(\Delta\) be complete and polytopal, and let \(-P^*\) be a spanning polytope of \(\Delta\), so that \(P\) is an associated polytope of \(\Delta\). Then, for vert \(P = \{a_1, \ldots, a_q\}\),

\[
P = \mathcal{P}(P) = [a_1 + \text{pos} \{P - a_1\}, \ldots, a_q + \text{pos} \{P - a_q\}]
\]

is an element of \(\text{Pic} \Delta\) from which \(\Delta\) can be reconstructed. Thus \(\Delta = \Delta(-P)\) for \(P = (a_1 + \text{pos} \{P - a_1\}) \cap \ldots \cap (a_q + \text{pos} \{P - a_q\})\).

Any summand of \(P^*\) and \(P\) can also be written in the form

\[
P' = (a'_1 + \text{pos} \{P - a_1\}) \cap \ldots \cap (a'_q + \text{pos} \{P - a_q\})
\]

where the assignment \(a_i \mapsto a'_i\) provides a surjective map \(\chi_{P'} : \text{vert} P \to \text{vert} P'\).
Definition 5. If $P'$ is a lattice summand of an associated polytope $P$ of the polytopal fan $\Delta$, we call
\[ \mathcal{P}(P') := [a'_1 + \text{pos}(P - a_1), \ldots, a'_q + \text{pos}(P - a_q)] \]
a polytope element of $\text{Pic} \Delta = \text{Pic} \Delta(-P)$.

Lemma 6. Let $\Delta = \Delta(-P)$ and let $P', P''$ be lattice polytopes such that $P = P' + P''$. Then
\[ \mathcal{P}(P) = \mathcal{P}(P') + \mathcal{P}(P''). \]
In particular for any $r \in \mathbb{N}$, $\mathcal{P}(rP) = r\mathcal{P}(P)$.

The following theorem (Ewald 1996, pp.175-7) enables us to find a finite system of generators of $\text{Pic} \Delta(-P)$, consisting of polytope elements.

Theorem 3. For any $P \in \text{Pic} \Delta(-P)$ there exists a lattice polytope $P_0$ strictly combinatorially isomorphic to $P$ (hence also associated with $\Delta$), and a natural number $r$ such that
\[ P = \mathcal{P}(P_0) - \mathcal{P}(rP). \]

Definition 6. If $\Delta = \Delta(P)$ is polytopal, we call the group $\mathcal{G}$ the polytopal group of $\Delta$.

The following result follows immediately.

Theorem 4. Let $\Delta = \Delta(P)$ be polytopal.

(i) The polytope group $\mathcal{G}$ is the smallest group into which the semigroup of all polytopes strictly combinatorially isomorphic to $P$ can be embedded.

(ii) $\text{Pic} \Delta$ can be generated by $f_{n-1}(P) - n - \lambda + 1$ polytope elements strictly isomorphic to $P$.

References