Imposing planes on 3-folds

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General themes
Classification in projective algebraic geometry (equiv: in finitely-generated graded rings).

Seek **families** of graded rings with particular properties.

Complicated families can be inferred from the existence of special elements in easy families.

- What is 'imposing a plane'?
- Line in cubic surface: conic fibrations
- Plane in quartic 3-fold: cubic surface fib'ns
- Magma analysis of Pfaffian equations.

Part I: Lines on cubic surfaces

The **family of cubic surfaces** is

$$S_3\colon (f_3 = 0) \subset \mathbb{P}^3$$

as $f_3 = f_3(x_0, x_1, x_2, x_3)$ varies through homogeneous forms of degree 3
(for which $(f_3 = 0)$ is nonsingular).

Fix a line $\mathbb{P}^1 \cong L = (x_0 = x_1 = 0) \subset \mathbb{P}^3$.

Imposing L means considering the subfamily of the form

$$L \subset S_3 = (x_0 f - x_1 g = 0) \subset \mathbb{P}^3$$

for homogenous quadrics f, g (for which ...).

Such S is nonsingular along L iff

$$\nabla_{|L} = (f, -g, 0, 0) \quad \text{is never zero.}$$

So Bertini $\Longrightarrow \exists$ such nonsing cubic surfaces.

What good is a line?

Special surfaces have special structures:

$$L \subset S_3 = (x_0 f - x_1 g = 0) \subset \mathbb{P}^3$$

(a) conic fibration given by the map

$$(x_2 \colon x_3) \colon S_3 \longrightarrow \mathbb{P}^1.$$

(c.f. Cremona's talk last year on conic fibrations and Tsen's theorem.)

(b) new birational model by contracting L: set $\quad s = f/x_1 = g/x_0 \quad$ to get a new surface

$$T \colon (x_1 s = f, x_0 s = g) \subset \mathbb{P}^4.$$

(a) is a structure theorem for S; (b) builds a more complicated surface out of $L \subset S$:

$$\mathbb{P}^4 \supset T \quad \longleftarrow \quad S \quad \longrightarrow \quad \mathbb{P}^1.$$

Planes on quartic 3-folds

Consider imposing a plane as

$$\begin{aligned} \mathbb{P}^2 \cong E &= (x_0 = x_1 = 0) \\ &\subset X_4 = (x_0 f_3 - x_1 g_3 = 0) \subset \mathbb{P}^4. \end{aligned}$$

Such X is nonsingular along E iff

$$\nabla_{|E} = (f, -g, 0, 0) \quad \text{is never zero.}$$

But Bézout $\implies$ vanishing at 9 points.

In general, these sings are nodes: $xy = zt$.
(Thinks: local non-unique factorisation in X.)

So Bertini $\implies \exists\ X \supset E$ with nodes on E.

What good is a plane?

New constructions and/or structure theorems:

$$\mathbb{P}^5(1^5, 2) \supset Y_{3,3} \quad \dashleftarrow \quad X_4 \quad \dashrightarrow \quad \mathbb{P}^1$$

where the fibration is by cubic surfaces.

Part II: Making Fano 3-folds

I am interested in building new Fano 3-folds. (**Dan Ryder**'s talk is on the structure theorems. **Stephen Coughlan**'s talk is on the commutative algebra behind the calculations.)

A **Fano 3-fold** is X s.t. anticanonical divisor $-K_X$ is ample (+ conditions on singularities).

In practice $X \subset \mathbb{P}^N$ is Fano when it is defined by few equations of small degree*.

Idea: construct complicated Y by finding simple $X \supset E$. (e.g. with fewer variables)

Target: make a $Y \subset \mathbb{P}^7$ from $\mathbb{P}^2 \subset X \subset \mathbb{P}^6$.

The method is to make a new variable s having a pole along E. Conversely, given Y, recover X by eliminating the new variable s.

Difficulty: control the singularities of X.

*But writing them at random usually generates rubbish when $N \geq 6$

The Magma graded ring webpage

There is a super-classification of possible Fano 3-folds on the graded ring webpage at

`www.kent.ac.uk/ims/grdb/`

It includes

$$Y_{3,3,4,4,4,5,5,5,6} \subset \mathbb{P}^7(1,1,1,1,1,2,3,4)$$

(The 8 variables have weights $1, 1, \ldots, 4$ and Y is defined by 9 eqns of indicated hgs degs.) The task is to show that such a Fano exists.

The webpage indicates that eliminating the weight 4 variable gives

$$\mathbb{P}^2(1,1,3) \subset X_{3,3,4,4,4} \subset \mathbb{P}^6(1,1,1,1,1,2,3).$$

I cannot write down equations for Y, but using Magma I can try to do it for X.

Part III: Analysing Pfaffians

A 3-fold $X \subset \mathbb{P}^6$ might be defined by 3 equations. But in this case, it is defined by 5 equations, the maximal Pfaffians of a skew 5×5 matrix. It has 10 indep entries

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{pmatrix}$$

(with 0s on diagonal, and skew-symm).
The Pfaffians are expressions like

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}$$

and similarly for 12.35, 12.45, etc.

In our case, the 10 entries have hgs degs

$$\begin{pmatrix} 3 & 2 & 2 & 2 \\ & 2 & 2 & 2 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Putting general forms of these degs gives a nice Fano 3-fold X but **without** $E \subset X$.

Imposing a plane

There are two known formats for imposing a plane $E \subset X$: codenames Tom & Jerry*.

Tom :
$$\left(\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \hline & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{array}\right)$$
entries below line must lie in I_E

Jerry :
$$\left(\begin{array}{cc|cc} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{array}\right)$$
entries to right of line must lie in I_E

If you do either of these (or something symm), then the Pfaffians will define an X cont'g E.

There are then formulas, due to Papadakis, for writing down the Y. (Or use Coughlan's method following Kustin–Miller.)

*I hope somebody has a better idea one day.

Michael Kerber's method

Fix $\mathbb{P}^2 = E \subset \mathbb{P}^6$.

For all possible T & J configs in the matrix.

Repeat 20 times: fill in with fairly general poly ring entries according to the config and compute Pfaffians.

All eqns taken together determine locus where Bertini-like theorems cannot apply.

Analyse this 'base locus' separately.

In a general example, analyse sings on $E \subset X$ separately.

This happens every time you click **Type I** unprojection on the webpage.

The theorem proved by Magma

116 of 145 codim 4 Fanos $Y \subset \mathbb{P}^7$ on the webpage have a Type I projection.

Tom and Jerry lead to **different** Y.

In every case there are 5 configs for Tom and 10 configs for Jerry.

Kerber + Magma prove that each of the 116 cases has at least one Tom and at least one Jerry subfamily. A little more work gives:

Theorem/Conjecture
For fixed Riemann–Roch data (+ . . .), the Hilbert scheme (which parametrises ideals in the given variables with the given Hilbert poly invariants) always has at least two components that parametrise ideals of Fano 3-folds.