COMPUTATIONAL FORMAL RESOLUTION OF SURFACES IN $\mathbb{P}^3_{\mathbb{C}}$

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Abstract

In Algebraic Geometry smooth varieties are (in general) well-understood. By contrast (or simply because of that) the objects of interest are often singular varieties. From the theoretical point of view a remedy for this situation is the famous Theorem of Hironaka [8] on the resolution of singularities: If X is a variety over a field of characteristic 0 then there always exists a smooth variety $\widetilde{X}$ and a proper birational morphism $\pi : \widetilde{X} \to X$. So for proving theorems and defining birational invariants, one can argue on $\widetilde{X}$ rather than on X and finally transfer the result back to the singular variety. This theorem has been made constructive by Villamayor [10], Bierstone-Milman [3] and others. There are also two implementations of the resolution algorithm in Singular, one by Anne Frübis-Krüger [6] and another by Gábor Bodnár and the second author [4]. In principal this makes many theoretical results algorithmic, but any algorithm relying on a resolution suffers from the high computational complexity of the resolution process.

From the computational point of view it is not always necessary to describe the resolution completely. In the case of algebraic curves over a field $\mathbb{K}$ series expansion have proved to be an important algorithmic tool. Here the preimage of the singular locus w.r.t. a resolution of singularities is a finite set of points. The idea is to describe the resolution by power series expansions that determine analytic (or formal) neighborhoods of these points. If $\mathbb{K}$ has characteristic 0 Puiseux expansions can be used and the Newton-Puiseux algorithm is implemented in many computer algebra systems including Magma, Maple and Singular. The latter system also contains an implementation of Hamburger-Noether expansions [5], that provide a similar tool in positive characteristic. Applications are for example the computation of integral bases in the function field [9], the computation of linear systems of divisors [7] and as a special case systems of adjoint curves used for parametrization [2].

To our knowledge for higher dimensional varieties there is no similar tool (at least no accessible implementation). The purpose of this talk is to present such a tool for surfaces in $\mathbb{P}^3_{\mathbb{C}}$ and its implementation in Magma. This is achieved by the following means.

Date: June 30, 2006.

The authors were supported by the FWF (Austrian Science Fund) in the frame of the research projects SFB 1303 and P15551.

The analog to the Newton-Puiseux Theorem in the multivariate case is the Theorem of Jung-Abhyankar. It states that if a variety is defined by the vanishing of a quasi-ordinary polynomial (i.e. a monic polynomial s.t. its discriminant defines a normal crossing divisor at the origin) then all the roots of the polynomial can be expressed as multivariate fractionary power series. These roots describe analytic isomorphisms from points in the normalization to germs of toric varieties. Now the first step is to develop an algorithm for the efficient computation and representation of these roots in finite terms, see [1]. In other words the algorithm takes as input a quasi-ordinary polynomial and produces finite representations of the series roots that may be expanded later up to arbitrary precision.

The next step is to find an analogue to the finitely many places over the singular points of an algebraic curve and their representation by Puiseux series. Given a singular surface X we show how to compute a finite set of embeddings $\kappa_i : \mathcal{K}_X \to \mathbb{K}_i((t))$ from the function field of the surface into fields of Laurent series. Here $\mathbb{K}_i$ is the function field of some algebraic curve. This set has the property that there is a resolution $\pi : \widetilde{X} \to X$ and the following hold true:

- The preimage $\kappa_i^{-1}(\mathbb{K}_i[[t]])$ is isomorphic to the completion of the local ring at a curve $C \subset \widetilde{X}$ that we call the center of κ_i.
- For every curve C in the π-preimage of the singular locus there is i s.t. C is the center of κ_i.

Such a set of embeddings is not unique as in the case of algebraic curves, but it should describe the singular locus reasonably well.

A brief sketch of the algorithm: Using the Noether normalization trick, we understand the surface as a finite cover of the projective plane $\mathbb{P}^2_{\mathbb{C}}$. Then we compute an embedded resolution of the discriminant curve by point blow ups and transform the surface accordingly. For determining the embeddings we compute power series roots of the local defining equations in the above form. We distinguish two cases: Over the generic points of irreducible curves of the discriminant we use the normal Newton-Puiseux algorithm to find such roots directly. Over the normal crossing locus of the discriminant we first compute fractionary power series roots and then transform each of them into finitely many series in the desired form.

An application is for example the computation of the sheaf of adjoint differential forms over a singular surface X. This sheaf (resp. the pushforward w.r.t. a birational morphism) is a birational invariant of a surface and corresponds to the sheaf of regular differential forms in the case of a smooth surface. First tests with the algorithm show that it performs very well.

References

[1] T. Beck and J. Schicho. Approximate Roots in Graded Rings. Technical Report 2005-03, RICAM, March 2005.

[2] Tobias Beck and Josef Schicho. Parametrization of Algebraic Curves Defined by Sparse Equations. *AAECC*, 2006. Accepted for publication.

[3] Edward Bierstone and Pierre D. Milman. A simple constructive proof of canonical resolution of singularities. In *Effective methods in algebraic geometry (Castiglioncello, 1990)*, volume 94 of *Progr. Math.*, pages 11–30. Birkhäuser Boston, Boston, MA, 1991.

[4] Gábor Bodnár and Josef Schicho. A computer program for the resolution of singularities. In *Resolution of singularities (Obergurgl, 1997)*, volume 181 of *Progr. Math.*, pages 231–238. Birkhäuser, Basel, 2000.

[5] A. Campillo and J. I. Farrán. Symbolic Hamburger-Noether expressions of plane curves and applications to AG codes. *Math. Comp.*, 71(240):1759–1780 (electronic), 2002.

[6] A. Frühbis-Krüger and G. Pfister. Practical Aspects of Algorithmic Resolution of Singularities. In *Reports On Computer Algebra*, number 33. Centre for Computer Algebra, University of Kaiserslautern, October 2004. Online available at http://www.mathematik.uni-kl.de/ zca.

[7] Gaétan Haché and Dominique Le Brigand. Effective construction of algebraic geometry codes. *IEEE Trans. Inform. Theory*, 41(6, part 1):1615–1628, 1995. Special issue on algebraic geometry codes.

[8] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[9] Mark van Hoeij. An algorithm for computing an integral basis in an algebraic function field. *J. Symbolic Comput.*, 18(4):353–363, 1994.

[10] Orlando Villamayor. Constructiveness of Hironaka's resolution. *Ann. Sci. École Norm. Sup. (4)*, 22(1):1–32, 1989.

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