Lattices and spherical designs.

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MAGMA conference, Berlin, 1 August 2006

Definition. (Delsarte, Goethals, Seidel 1977)
A finite nonempty set

$$X \subset S^{n-1} = \{x \in \mathbb{R}^n \mid (x,x) = 1\}$$

is a **spherical t-design** if for all polynomials f of degree $\leq t$

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \int_{S^{n-1}} f(x) d\mu(x).$$

Since $f_{\alpha,k} : x \mapsto (x,\alpha)^k$ with $\alpha \in \mathbb{R}^n$ generate the space of homogeneous polynomials of degree k, this is equivalent to asking

$$\sum_{x \in X} (\alpha, x)^k = |X| \int_{S^{n-1}} (\alpha, x)^k d\mu(x) = \begin{cases} 0 & k \text{ odd} \\ c_k |X| (\alpha,\alpha)^{k/2} & k \text{ even} \end{cases}$$

where $c_k = \prod_{j=1}^{k/2} \frac{2j-1}{n+2j-2}$ for $k = 1, \ldots, t$.

If $X \subset S^{n-1}$ is a spherical t-design, then

$$|X| \geq \binom{n-1+t/2}{t/2} + \binom{n-2+t/2}{t/2-1} \quad t \text{ even}$$
$$|X| \geq 2\binom{n-1+(t-1)/2}{(t-1)/2} \quad t \text{ odd}$$

If equality holds, then X is called a **tight t-design**.

Tight t-designs in S^{n-1} with $n \geq 3$ only exist for $t \leq 5$ or $t = 7, 11$. They are classified completely for $t \in \{1, 2, 3, 11\}$ and for $t = 4, 5, 7$ up to dimension $n = 104$.

Examples:

$t = 1$: $|X| = 2\binom{n-1}{0} = 2$, $X = \{x, -x\}$

$t = 2$: $|X| = n + 1$, **simplex**.

$t = 3$: $|X| = 2\binom{n}{1} = 2n$, $X = \{\pm e_1, \ldots, \pm e_n\}$
for ON-basis $(e_1, \ldots, e_n)$ (**cross polytope**)

$t = 7$: $n = 8$ and $X = \text{Min}(E_8)$, $|X| = 240$.

$t = 7$: $n = 23$ and $X = \text{Min}(O_{23})$, $|X| = 4600$.

$t = 11$: $n = 24$ and $X = \text{Min}(\Lambda_{24})$, $|X| = 196560$.

Definition. A **lattice** $L \subset \mathbb{R}^n$ is the set of all integral linear combinations of a basis

$$L = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}}.$$

The **dual lattice** is

$$L^* = \{\alpha \in \mathbb{R}^n \mid (\alpha, x) \in \mathbb{Z} \text{ for all } x \in L\}.$$

The **minimum** of L is

$$\min(L) = \min\{(x,x) \mid 0 \neq x \in L\}$$

and we denote by

$$\mathrm{Min}(L) := \{x \in L \mid (x,x) = \min(L)\}$$

the **set of minimal vectors** in L.

L is called a **t-design lattice**, if $\mathrm{Min}(L)$ forms a spherical t-design and generates L.

Fact. Let $X = -X \subset S^{n-1}$ be antipodal.
Then X is a $2h$-design $\Leftrightarrow$ X is a $2h+1$-design.
Proof. f homogeneous of odd degree, then $f(-x) = -f(x)$ hence

$$\sum_{x \in X} f(x) = 0$$

for any antipodal set X.

Corollary. L is $2h$-design lattice $\Leftrightarrow$ L is $2h+1$-design lattice.

The 5-design lattices L of dimension $n \leq 12$

n	L	t	$\|\mathrm{Min}(L)\|$	$\min(L)\min(L^*)$	tight
1	$\mathbb{Z}$	∞	2	1	
2	A_2	5	6	2/3	yes
4	D_4	5	24	1	
6	E_6	5	72	8/3	
6	E_6^*	5	54	8/3	
7	E_7	5	126	3	
7	E_7^*	5	56	3	yes
8	E_8	7	240	4	yes
10	K'_{10}	5	270	4	
10	K'^*_{10}	5	240	4	
12	K_{12}	5	756	16/3	

7-design lattices L of dimension $n \leq 24$
(List complete (for $n \neq 23$))

n	L	t	$\lvert \mathrm{Min}(L) \rvert$	$\min(L)\min(L^*)$	tight
1	$\mathbb{Z}$	∞	2	1	
8	E_8	7	240	4	yes
16	Λ_{16}	7	4320	8	
23	O_{23}	7	4600	9	yes
23	Λ_{23}	7	93150	12	
24	Λ_{24}	11	196560	16	yes

No t-design lattices are known for $t \geq 12$.

Theorem. 5-**design lattices are local maxima for the density function.**

The theory of designs provides tools to classify t-design lattices of small dimension and hence a method to find certain nice dense lattices.

Note. Local maxima for the density function are similar to rational lattices. In particular t-design lattices are rational if $t \geq 4$.

The Hermite function γ is a positive function on the space of similarity classes of n-dimensional lattices defined by

$$\gamma(L) = \frac{\mathrm{min}(L)}{\mathrm{det}(L)^{1/n}}$$

with $\mathrm{det}(L) = \mathrm{vol}(L)^2$ the determinant of a Grammatrix of L.

$\gamma(L)$ is proportional to the density of the sphere packing associated with L.

γ has only finitely many local maxima which may be characterized as those lattices L that are perfect and eutactic
(Voronoi, Korkine, Zolotareff $\sim$ 1900).

A lattice L is **eutactic**, if there are $\lambda_x > 0$ ($x \in \mathrm{Min}(L)$) such that

$$(\alpha,\alpha) = \sum_{x \in \mathrm{Min}(L)} \lambda_x (x,\alpha)^2 \text{ for all } \alpha \in \mathbb{R}^n.$$

2-design lattices are eutactic with $\lambda_x = (c_2 |\mathrm{Min}(L)|)^{-1}$ for all $x \in \mathrm{Min}(L)$.
L is **perfect**, if the orthogonal projections

$$p_x : \alpha \mapsto (x,\alpha)x$$

along $x \in \mathrm{Min}(L)$ span the space of all symmetric endomorphisms of $\mathbb{R}^n$.
4-design lattices are perfect.
(proof quite similar to above).

Bounds on the minimum of t-design lattices.

The **Bergé-Martinet-invariant** of a lattice L is

$$\gamma'(L)^2 := \gamma(L)\gamma(L^*) = \min(L)\min(L^*).$$

Theorem. If $L \subset \mathbb{R}^n$ is a 4-design lattice, then $\gamma'(L)^2 \geq (n+2)/3$. If equality holds then $(\alpha, x) \in \{0, \pm 1\}$ for all $\alpha \in \mathrm{Min}(L^*)$ and $x \in \mathrm{Min}(L)$.

Proof. $D_4(\alpha) - D_2(\alpha) =$

$$\sum_{x \in \mathrm{Min}(L)} (x,\alpha)^2((x,\alpha)^2 - 1)) = \frac{|\mathrm{Min}(L)|(\alpha,\alpha)(x,x)}{n}((x,x)(\alpha,\alpha)\frac{3}{n+2} - 1).$$

For $\alpha \in L^*$ this is nonnegative since $(x, \alpha) \in \mathbb{Z}$ for $x \in \mathrm{Min}(L) \subset L$. Choosing $\alpha \in \mathrm{Min}(L^*)$ we find

$$(\alpha,\alpha)(x,x) = \min(L)\min(L^*) = \gamma'(L)^2 \geq (n+2)/3$$

and "$=$" $\Leftrightarrow D_4(\alpha) - D_2(\alpha) = 0 \Leftrightarrow (\alpha, x) \in \{0, \pm 1\} \forall x \in \mathrm{Min}(L)$.

General method: L a t-design lattice, $t = 2h$, $\alpha \in \mathrm{Min}(L^*)$. Then

$$\sum_{x \in \mathrm{Min}(L)} \prod_{j=0}^{h-1} ((x,\alpha)^2 - j^2) = \frac{|\mathrm{Min}(L)|\gamma'(L)^2}{n} P_{n,t}(\gamma'(L)^2) \geq 0$$

where $P_{n,t}(z)$ is a polynomial of degree $h-1 = t/2-1$ in $z = \gamma'(L)^2$.

For small t, the polynomials $P_{n,t}$ are as follows:

$$\begin{aligned}
P_{n,2}(z) &= 1 \\
P_{n,4}(z) &= \tfrac{3}{n+2}z - 1 \\
P_{n,6}(z) &= \tfrac{3\cdot 5}{(n+2)(n+4)}z^2 - 5\tfrac{3}{n+2}z + 4 \\
P_{n,8}(z) &= \tfrac{3\cdot 5\cdot 7}{(n+2)(n+4)(n+6)}z^3 - 14\tfrac{3\cdot 5}{(n+2)(n+4)}z^2 + 49\tfrac{3}{n+2}z - 36
\end{aligned}$$

Remark. Let L be a 6-design lattice of dimension $n > 1$. Then $\gamma'(L)^2 > \frac{n+2}{3}$,

Proof. If $\gamma'(L)^2 = \frac{n+2}{3}$ then $(\alpha, x) \in \{0, \pm 1\}$ for all $\alpha \in \mathrm{Min}(L^*)$, $x \in \mathrm{Min}(L)$. Hence $\frac{n+2}{3}$ is also a zero of $P_{n,6}(t)$ which implies that $5(n+2) = 3(n+4)$ whence $n = 1$.

For an 8-design lattice $L \leq \mathbb{R}^n$, we have $\gamma'(L)^2 \geq b(n)$ where $b(n)$ is the real root of $P_{n,8}(z)$.
For a 12-design lattice $L \leq \mathbb{R}^n$, we have $\gamma'(L)^2 \geq c(n)$ where $c(n)$ is the real root of $P_{n,12}(z)$.

n	26	32	36	40	48	50	66	72	80
$\frac{n+2}{3}$	9.33	11.33	12.66	14	16.66	17.33	22.66	24.66	27.33
$b(n)$	16	20.66	24	27.49	34.38	36	48	52.31	58.01
$c(n)$	17.88	23.35	27.24	31.16	38.54	40.29	53.64	58.53	64.99

The **Hermite constant** $\gamma_n := \max\{\gamma(L) \mid L \leq \mathbb{R}^n\}$ satisfies

$$\frac{1}{2\pi e} \leq \frac{\gamma_n}{n} \leq \frac{1.744}{2\pi e}$$

The best bound for $\gamma'(L)^2$ currently available is

$$\gamma'(L)^2 \leq \gamma_n^2 \sim n^2.$$

If n tends to infinity then the real roots of $P_{n,t}$ are approximately n, yielding

$$n \lesssim \gamma'(L)^2 \lesssim n^2$$

for a t-design lattice $L \leq \mathbb{R}^n$ which does not give a contradiction to the possible existence of such lattices for arbitrarily big t.

Towards a classification of t-design lattices.

Let L be a t-design lattice with $t = 2h$ even. For $\alpha \in L^*$ and $i \in \mathbb{N}$ put

$$N_i(\alpha) := \{x \in \mathrm{Min}(L) \mid (x, \alpha) = i\}.$$

If $N_i(\alpha) = \emptyset$ for all $i > h$ then

$$|N_h(\alpha)| = \frac{1}{h \cdot (t-1)!} \sum_{x \in \mathrm{Min}(L)} \prod_{j=0}^{h-1} ((x, \alpha)^2 - j^2) =$$

$$\frac{|\mathrm{Min}(L)|(\alpha, \alpha)(x, x)}{nh \cdot (t-1)!} P_{n,t}((\alpha, \alpha)(x, x))$$

and there are similar expressions for $|N_i(\alpha)|$ for $0 \leq i \leq h$.

Theorem. If $(\alpha, x) \in \{0, \pm 1, \ldots, \pm h\}$ for all $x \in \mathrm{Min}(L)$ then

$$\sum_{x \in N_h(\alpha)} x = \frac{|N_h(\alpha)| h}{(\alpha, \alpha)} \alpha.$$

Proof. For $\gamma \in \mathbb{R}^n$

$$\sum_{x \in \mathrm{Min}(L)} (x, \gamma)(x, \alpha) \prod_{j=1}^{h-1} ((x, \alpha)^2 - j^2) = c \sum_{x \in N_h(\alpha)} (x, \gamma) = c'(\alpha, \gamma).$$

This implies that $\sum_{x \in N_h(\alpha)} x = \frac{c'}{c} \alpha$ where one gets the constant by taking the scalar product with α.

Let $X \subset S^{n-1}$ be a spherical t-design then for all $k \leq t$ and all $\alpha \in \mathbb{R}^n$

$$(D_k)(\alpha): \quad \sum_{x \in X} (\alpha, x)^k = \begin{cases} 0 & k \text{ odd} \\ c_k |X| (\alpha, \alpha)^{k/2} & k \text{ even} \end{cases}$$

where $c_k = \prod_{j=1}^{k/2} \frac{2j-1}{n+2j-2}$.

Substituting $\alpha = \xi_1 \alpha_1 + \xi_2 \alpha_2$ in (D_k) and comparing coefficients we find that for all $i, j \in \mathbb{N}$ with $i + j \leq t$ there is a polynomial $p_{i,j}$ such that for all $\alpha, \beta \in \mathbb{R}^n$

$$(D_{ij})(\alpha, \beta): \quad \sum_{x \in X} (x, \alpha)^i (x, \beta)^j = p_{i,j}((\alpha, \alpha), (\beta, \beta), (\alpha, \beta))$$

Theorem. Let L be a t-design lattice with $t = 2h$ even and let $\alpha \in L^*$ such that $(\alpha, x) \in \{0, \pm 1, \ldots, \pm(h-d)\}$ for all $x \in \mathrm{Min}(L)$. Then the projection of $N_{h-d}(\alpha)$ onto $\alpha^\perp$ is a $2d+1$-design.

Proof. (idea) For $j \in \{0, \ldots d\}$ consider

$$\sum_{x \in N_{h-d}(\alpha)} (x, \beta)^{2j} = c \sum_{x \in \mathrm{Min}(L)} \prod_{i=0}^{h-d-1} ((x, \alpha)^2 - i^2)(x, \beta)^{2j}$$

which is a linear combination of the $p_{2\ell, 2j}$ with $\ell + j \leq h$.

Corollary. Let $L \subset \mathbb{R}^n$ be a 6-design lattice with $\gamma'(L)^2 = 8$ scaled such that $\min(L) = 2$, $\min(L^*) = 4$.
Then $n = 16$ and for all $\alpha \in \mathrm{Min}(L^*)$
$N_2(\alpha) = \{x_i, \alpha - x_i \mid 1 \leq i \leq 15\}$ and $\langle N_2(\alpha), \alpha \rangle \cong D_{16}$.

Proof. $\alpha \in \text{Min}(L^*)$, $x \in \text{Min}(L) \Rightarrow (\alpha, x) \in \{0, \pm 1, \pm 2\}$.
Hence $P_{n,6}((\alpha,\alpha)(x,x)) = P_{n,6}(8) = 0$ which yields $n = 8$ or $n = 16$. Since $\gamma_8 = 2$ the only possibility is that $n = 16$.

For $x \in N_2(\alpha)$ let $\overline{x} := x - \frac{2}{(\alpha,\alpha)}\alpha \in \alpha^{\perp}$. Then for all $x, y \in N_2(\alpha)$ we get

$$(\overline{x}, \overline{y}) = (x, y) - 1 = \begin{cases} 1 & x = y \\ \leq 0 & x \neq y \end{cases}$$

since x and y are minimal vectors of a lattice. Hence $\overline{N_2(\alpha)}$ is a set of vectors of length 1 in an $n-1$-dimensional space such that distinct vectors have non-positive inner products. Therefore $|\overline{N_2(\alpha)}| \leq 2(n-1)$. Since $N_2(\alpha)$ is a 3-design, we find that $|N_2(\alpha)| \geq 2(n-1)$ and hence equality holds and $\overline{N_2(\alpha)}$ is a cross polytope ($(\overline{x}, \overline{y}) = 0$ if $\overline{x} \neq \pm\overline{y}$).

This gives the Grammatrix of $N_2(\alpha)$.

Grammatrix for $(x_1, \ldots, x_{15}, \alpha)$

$$\begin{pmatrix} 2 & 1 & \ldots & 1 & 2 \\ 1 & 2 & \ldots & 1 & 2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 2 & 2 \\ 2 & 2 & \ldots & 2 & 4 \end{pmatrix}$$

Theorem. **The 16-dimensional 6-design-lattices are similar to the Barnes-Wall lattice.**

Proof follows from

Lemma. Let L be a 6-design lattice of dimension 16. Then $\gamma'(L)^2 = 8$.

together with the Corollary above:

Corollary. Let $L \subset \mathbb{R}^n$ be a 6-design lattice with $\gamma'(L)^2 = 8$ Then $n = 16$ and for all $\alpha \in \mathrm{Min}(L^*)$

$$L \supset \langle N_2(\alpha), \alpha \rangle \cong D_{16}.$$

General strategy: If L is a $2h$-design lattice and $\alpha \in \mathrm{Min}(L^*)$ then for $1 \leq j \leq h$

$$\sum_{x \in \mathrm{Min}(L)} (x,\alpha)^{2j} = \gamma'(L)^{2j} |\mathrm{Min}(L)| \prod_{k=1}^{j} \frac{2k-1}{n-2k+2} \in 2\mathbb{Z}$$

$$\frac{1}{(2h-1)!} \sum_{x \in \mathrm{Min}(L)} \prod_{j=0}^{h-1} ((x,\alpha)^2 - j^2) = \frac{|\mathrm{Min}(L)|\gamma'(L)^2}{n} P_{n,t}(\gamma'(L)^2) \in 2\mathbb{Z}$$

and

$$\frac{1}{h!^2} \sum_{x \in \mathrm{Min}(L)} (x,\alpha)^2 \prod_{j=1}^{(h-1)/2} ((x,\alpha)^2 - j^2)^2 = |\mathrm{Min}(L)| p_{n,h}(\gamma'(L)^2) \in 2\mathbb{Z}$$

are even non-negative integers.

Proof of Lemma. Put $r := \gamma'(L)^2$. Then
$r \in \mathbb{Q}$ and $r \leq \gamma_{16}^2 \leq 9.163$.
Bounds on kissing numbers yield $s := |\operatorname{Min}(L)| \leq 8160$.
For $\alpha \in \operatorname{Min}(L^*)$ the sum

$$\frac{1}{12} \sum_{x \in \operatorname{Min}(L)} (\alpha, x)^2((\alpha, x)^2 - 1) = \frac{sr}{12 \cdot 16}(\frac{r}{6} - 1)$$

is a positive integer $\leq s\frac{\gamma_{16}^2}{12 \cdot 16}(\frac{\gamma_{16}^2}{6} - 1) \leq 0.0252s$.
Going through all possibilities for

$$(s, a) \in [1632, 8160] \times [1, 0.0252s]$$

using the fact that r is a positive rational solution of

$$\frac{sr}{12n}(\frac{3r}{n+2} - 1) - a = 0$$

satisfying integrality conditions above (with $h = 3$) the only possibility is $r = 8$.