Free resolutions, birational geometry and unprojection

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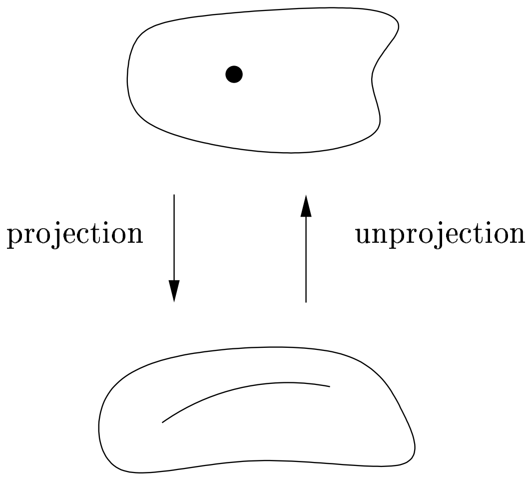
Overview

Unprojection is the reverse operation to a projection. Take a suitable point on a variety and project from it. We call this point the centre of projection. The image of this projection will contain a divisor which is the image of the centre.



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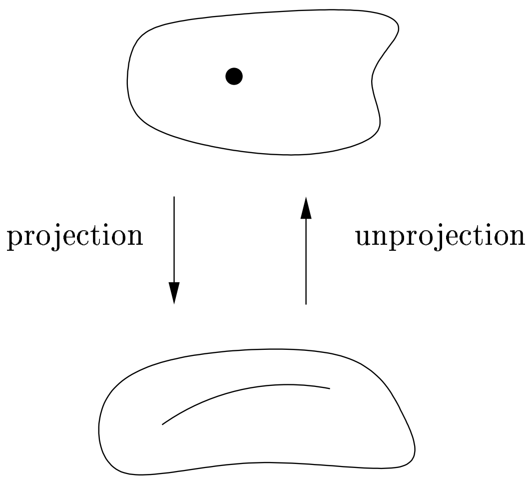

Unprojection contracts this divisor by finding rational functions with poles on D and adjoining these to the coordinate ring.

There are three issues here:

1. How to construct this special divisor, and the variety X containing it?
2. How to find rational functions with poles on this divisor?
3. How to write down the relations defining the resulting variety Y?

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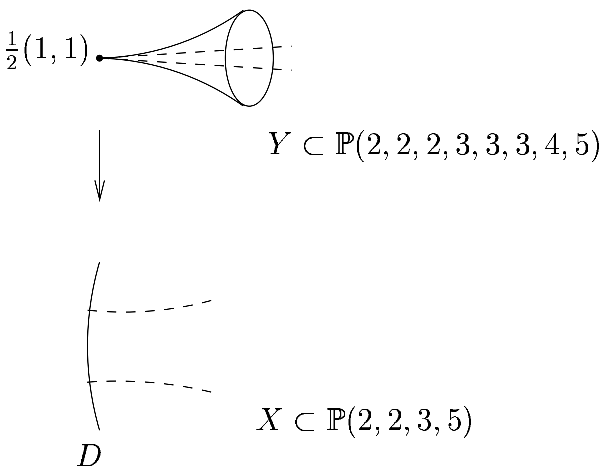
The example

We illustrate these problems and some solutions with a particular example from the K3 database, which asserts:

- there is a K3 surface $Y \subset \mathbb{P}(2,2,2,3,3,3,4,5)$, with singularities $7 \times \frac{1}{2}(1,1)$ and $\frac{1}{5}(2,3)$

which projects from a $\frac{1}{2}(1,1)$ point to

- K3 hypersurface $X_{12} \subset \mathbb{P}(2,2,3,5)$ containing $D \cong \mathbb{P}^1$, singularities $6 \times \frac{1}{2}(1,1)$ and $\frac{1}{5}(2,3)$.



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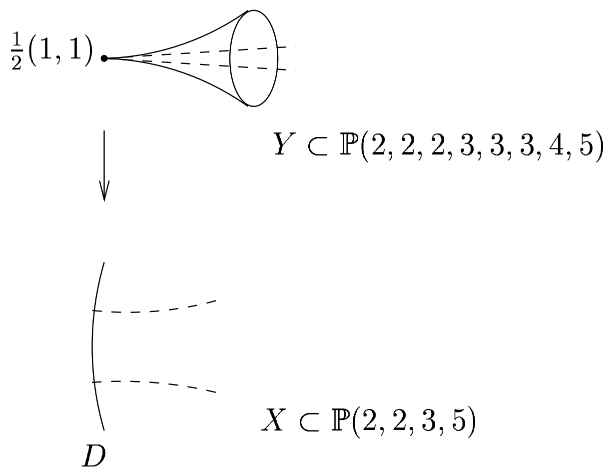

Step 1: Constructing $D \subset X_{12} \subset \mathbb{P}(2,2,3,5)$

The first problem we address is the construction of the hypersurface X containing a copy of $\mathbb{P}^1$.

We want to write down an embedding

$$\varphi : \mathbb{P}^1 \to D \subset \mathbb{P}(2,2,3,5).$$

Equivalently,

$H^0(\mathbb{P}(2,2,3,5), \mathcal{O}(n))$ must span $H^0(\mathbb{P}^1, \mathcal{O}(n))$ for all $n \gg 0$.

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Compare the Hilbert series of $\mathbb{P}^1$ with that of $\mathbb{P}(2,2,3,5)$:

n	$h^0(\mathbb{P}^1, \mathcal{O}(n))$	$h^0(\mathbb{P}(2,2,3,5), \mathcal{O}(n))$
0	1	1
1	2	0
2	3	2
$\vdots$	$\vdots$	$\vdots$
10	11	12
11	12	12
12	13	17
$\vdots$		

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One embedding suggested by MAGMA is:

$$(u, v) \mapsto (u^2, v^2, u^3 + u^2 v + uv^2, uv^4 + v^5).$$

Now choose a relation of degree 12 containing the image of φ. This is an easy elimination calculation.

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Resolution of $k[u,v]$

Let $\mathcal{O} = k[x,y,z,t]$, $k[X]$, $k[D]$ be the homogeneous coordinate rings of $\mathbb{P}(2,2,3,5)$, X and D respectively.

- D is not projectively normal - normalisation of $k[D]$ is $k[u,v]$ by construction.
- φ makes $k[u,v]$ into an $\mathcal{O}$-module generated by 1, u, v and uv.

For example:

$$u^2 = x.1 \quad v^2 = y.1 \quad u^3 = x.u$$

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$$k[u,v] \leftarrow 4\mathcal{O} \xleftarrow{M_1} 8\mathcal{O} \xleftarrow{M_2} 4\mathcal{O} \leftarrow 0,$$

which fits into a diagram with the resolution of $k[X]$ as an $\mathcal{O}$-module as follows:

$$\begin{array}{ccccccccc}
k[X] & \leftarrow & \mathcal{O} & \xleftarrow{f_{12}} & \mathcal{O}(-12) & & \leftarrow & 0 & \\
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[Kustin-Miller] and [Papadakis-Reid] showed that the generators of the last module $4\mathcal{O}$ are the unprojection variables. Call these s_0, s_1, s_2, s_3.

The s_i must have the same image under M_2 as the generator f_{12} of $\mathcal{O}(-12)$ has under the downarrow D.

We get relations $M_2(s_0, s_1, s_2, s_3) = D$, which are linear in the s_i.

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Step 3: Quadratic relations

We have not yet described the unprojected surface Y.

There should be further relations which are quadratic in the s_i, corresponding to the structure of $k[u,v]$ as an $\mathcal{O}$-module.

We expect relations like

$$(uv)(uv) - (u^2.1)(v^2.1) = 0 \leftrightsquigarrow s_3^2 - xys_0^2 = \dots$$

$$u.v - 1.uv = 0 \leftrightsquigarrow s_1 s_2 - s_0 s_3 = \dots$$

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$$s_3^2 - xys_0^2 - 9y^2s_3 - 3/2y^3s_0 + 27/2y^4 + 9/2xz^2 - \ldots$$

$$s_2s_3 - ys_0s_2 + ys_0s_1 - 5y^2s_2 - y^2s_1 - 13/2y^2z - \ldots$$

$$s_1s_3 + 6yt - ys_0s_2 + ys_0s_1 - 1/2y^2s_2 - 4y^2s_1 - \ldots$$

$$s_2^2 - 2s_0s_3 - 2ys_3 - ys_0^2 + 2y^2s_0 + 9/2y^3 - xs_0^2 + \ldots$$

$$s_1s_2 - s_0s_3 + 3ys_3 - ys_0^2 - y^2s_0 - 21/2y^3 + 8xs_3 + \ldots$$

$$s_1^2 - 16z^2 + 8ys_3 - ys_0^2 - 11/2y^2s_0 - 39y^3 + 8xs_3 + \ldots$$

$$ts_3 + z^3 - 6y^2t - y^3s_2 + y^3s_1 + 2y^3z - xyt - xy^2s_1 - \ldots$$

$$ts_2 + yz^2 - 2y^2s_3 - y^3s_0 + 3y^4 + 2xz^2 - xy^2s_0 - x^2y^2 \ldots$$

$$ts_1 + 4zt + yz^2 - y^2s_3 - y^3s_0 + xz^2 + xy^3 - 2x^2y^2 - \ldots$$

$$ts_0 + yt - y^2s_2 + 2xyz + 2x^2z$$

$$zs_3 - 3/2yt - 3y^2z - 9/2xt - xys_2 - 2xyz - x^2s_1 - 2x^2z$$

$$zs_2 - ys_3 + 3y^3 - 2xs_3 - 2xys_0 + 2xy^2 - x^2s_0 + x^2y$$

$$zs_1 + 4z^2 - ys_3 + 9/2y^3 - xs_3 - xys_0 + 3/2xy^2 + x^2y$$

$$zs_0 - ys_2 + ys_1 + 5yz - xs_2$$

$$z^4 + 3/2yt^2 - 3y^2zt - 2y^3z^2 - 3/2y^6 + 9/2xt^2 + xyzt + \ldots$$