Computational Formal Resolution of Surfaces in $\mathbb{P}^3_{\mathbb{C}}$

{Tobias.Beck|Josef.Schicho}@oeaw.ac.at

RICAM-Linz
Austrian Academy of Sciences

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Outline

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Gradings on Polynomial Rings

Let $<$ be a total ordering on $\mathbb{N}^2$ **compatible** with the **monoid structure** and with **degree**. We write $<$ also for the ordering induced on monomials, i.e. $x_1^{\mu_1} x_2^{\mu_2} < x_1^{\nu_1} x_2^{\nu_2}$ if $\mu < \nu$ for $\mu, \nu \in \mathbb{N}^2$.

$\mathbb{C}[x_1, x_2]$ becomes an $\mathbb{N}^2$-graded ring:

$$\mathbb{C}[x_1, x_2] = \bigoplus_{\mu \in \mathbb{N}^2} \mathbb{C} x_1^{\mu_1} x_2^{\mu_2}$$

Each $\mu \in \mathbb{N}^2$ induces an $\mathbb{N}^2$-grading on $\mathbb{C}[x_1, x_2][z]$ by setting

$$\deg_\mu(z) := \mu$$

which we call the μ**-induced grading**. Let IF_μ be the **initial form** in the μ-induced grading.

Example

Let $p := z^2 + (-x_1^2x_2^6 - 3x_1^4x_2^6 - 2x_1^2x_2^8 - x_1^4x_2^8)$ and $\mu := (1,3)$. Then

$$\deg_\mu(z^2) = 2(1,3) = (2,6) \text{ and } \deg_\mu(x_1^2x_2^6) = (2,6)$$

and $\boxed{\mathrm{IF}_\mu(p) = z^2 - x_1^2x_2^6}$. This polynomial has roots $\pm x_1x_2^3$.

Now consider

$$p' := p(z + x_1x_2^3) = z^2 + z(2x_1x_2^3) + (-3x_1^4x_2^6 - 2x_1^2x_2^8 - x_1^4x_2^8)$$

Let $\mu' := (3,3)$, then

$$\deg_\mu(zx_1x_2^3) = (3,3) + (1,3) = (4,6) \text{ and } \deg_\mu(x_1^4x_2^6) = (4,6)$$

and $\boxed{\mathrm{IF}_{\mu'}(p') = 2x_1x_2^3z - 3x_1^4x_2^6}$. This polynomial is linear and has a unique root $\frac{3}{2}x_1^3x_2^3$.

The Newton Algorithm

Assume $p \in \mathbb{C}[x_1, x_2][z]$ is **squarefree** and has a **power series root**:

$$p(\alpha) = 0 \text{ for } \alpha = \sum_{\mu \in \mathbb{N}^2} \alpha_\mu x_1^{\mu_1} x_2^{\mu_2} \in \mathbb{C}[[x_1, x_2]]$$

Then $p(z + \alpha) \in \mathbb{C}[[x_1, x_2]][z]$ has a root 0. In other words it has **no constant coefficient** w.r.t. z. If we denote

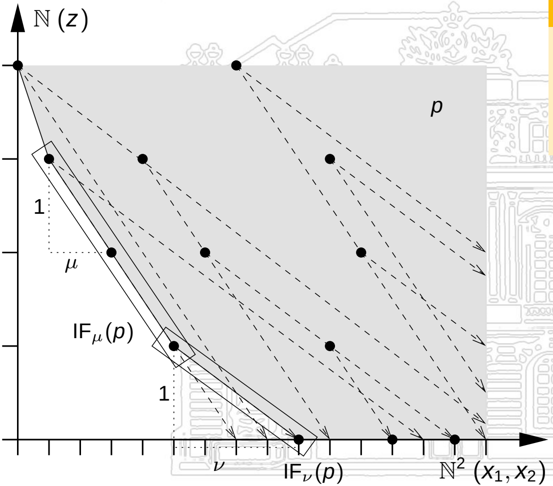
$$\alpha_{<\nu} := \sum_{\mu < \nu} \alpha_\mu x_1^{\mu_1} x_2^{\mu_2}$$

then the **order of the constant coefficient** of

$$p(x_1, x_2, z + \alpha_{<\nu}) \in \mathbb{C}[x_1, x_2][z]$$

increases for increasing ν.

The Newton Algorithm



Theorem

A power series α is a root of p iff α_μ is a root of $\mathrm{IF}_\mu(p(z+\alpha_{<\mu}))$.

Computing power series
=
solving univariate polynomial equations!

For **squarefree** p these equations are **eventually linear.**

Representation of Power Series

The polynomial p from the example has a root

$$\alpha = x_1x_2^3 + \frac{3}{2}x_1^3x_2^3 + x_1x_2^5 - \frac{9}{8}x_1^5x_2^3 - x_1^3x_2^5 - \frac{1}{2}x_1x_2^7 + \frac{27}{16}x_1^7x_2^3 + \dots$$

and α is represented by $(p, x_1x_2^3, (3,3)) = (p, a, \mu)$. This triple is s.t.:

$$\boxed{\mathrm{IF}_\mu(p(z+a)) \text{ has degree 1 in } z}$$

This point of view is particularly nice for implementation. Summing up, we can **effectively**

- represent multivariate algebraic power series,
- compute power series roots of polynomials (if they exist),
- carry out the ring operations for algebraic power series and
- deal with fractionary exponents.

Outline

Resolution of Varieties

Definition (Resolution of Varieties)

Let X be a variety over $\mathbb{C}$. A resolution of X is a **proper, birational** morphism $\phi : Y \to X$ where Y is a **smooth** variety.

proper: The ϕ-preimage of every compact subset is compact.

birational: The morphism ϕ has an “inverse” that can be described by rational functions.

Embedded Resolution of the Discriminant

Let $f \in \mathbb{C}[x, y, z, w]$ be an **irreducible, homogeneous** polynomial and $S \subset \mathbb{P}^3_{\mathbb{C}}$ the surface defined by the vanishing of f.

Locally S is a subset of $\mathbb{C}^3$ and defined by an **irreducible** polynomial $g \in \mathbb{C}[x_1, x_2][z]$. We can assume w.l.o.g. that g is also **monic** in z.

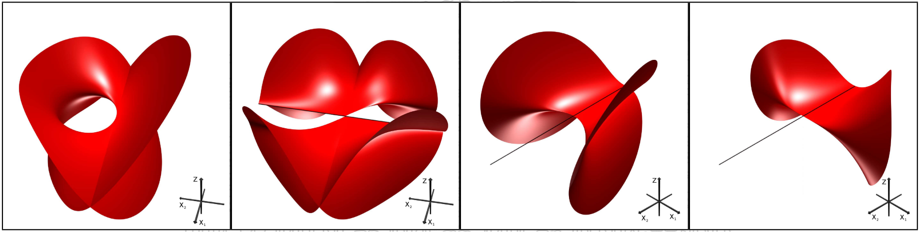
The polynomial $d := \mathrm{disc}_z(g)$ defines a curve $D \subset \mathbb{C}^2$ which we call the **discriminant curve**. It **bounds the singular locus** of S.

Theorem (Embedded Resolution of Curves)

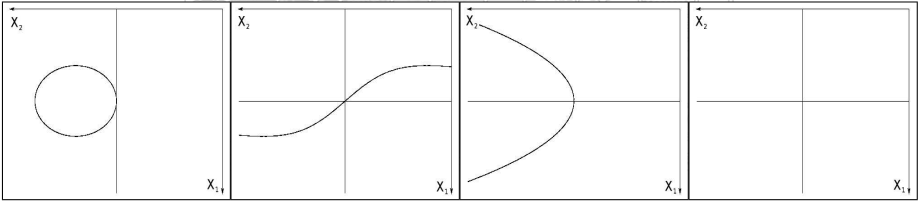
*After **finitely** many times **blowing up** the singular points of the discriminant curve, it has only **normal crossings**. I.e. all components are smooth and in each point at most two components of the curve intersect transversally.*

Embedded Resolution of the Discriminant

Surface



Discriminant Curve



Jung-Resolution

After these steps we have found a proper, birational map $\phi_0 : S_0 \to S$ s.t. S_0 is locally given by a **monic, irreducible** polynomial $g \in \mathbb{C}[x_1, x_2][z]$ and $\mathrm{disc}_z(g)$ defines a **discriminant** curve D_0 with **normal crossings**.

Theorem (Jung-Resolution)

Let $\phi_1 : S_1 \to S_0$ be the normalization of S_0. Let $\phi_2 : S_2 \to S_1$ be the morphism obtained by successively blowing up the singular points of S_1 (and its transforms). Then $\phi_0 \circ \phi_1 \circ \phi_2 : S_2 \to S$ is a resolution.

The normalization S_1 lives in a higher-dimensional space and involves Groebner basis computations. We want to

- “describe” the essential parts of S_2 while
- avoiding the explicit computation of S_1 and S_2 and
- staying in 3-space using only factorizations and linear algebra.

Outline

Definition of a Formal Resolution

Let $\phi : S_2 \to S$ be the Jung-Resolution, $\mathcal{S}_S \subset S$ the **singular locus** of S and $\mathcal{K}_S$ the **field of rational functions** on S.

Then $\mathcal{S}_S$ lies **above the discriminant curve** D. The preimage

$$\phi^{-1}(\mathcal{S}_S)$$

decomposes into **finitely many curves**, so-called **divisors** on S_2.

Definition (Formal Description of the Resolution of a Surface)

A **formal description of** ϕ is a **finite set of embeddings** $\kappa_i : \mathcal{K}_S \to \mathbb{F}_i((t))$ where $\mathbb{F}_i$ is a field of transcendence degree 1 over $\mathbb{C}$. For each curve C in $\phi^{-1}(\mathcal{S}_S)$ there must be an i s.t. the restriction of κ_i to $\kappa_i^{-1}(\mathbb{F}_i[[t]])$ **parametrizes a formal neighborhood of** C.

Example Application: Adjoint Differential Forms

Let $g := z^3 x_2^2 + (x_1 + z^2)^3$ be the defining equation of a surface $S \subset \mathbb{C}^3$. One of the computed embeddings in this case is

$$\kappa : \mathrm{Frac}(\mathbb{C}[x_1, x_2, z]/\langle g \rangle) \to \mathbb{F}((t)) : \begin{cases} x_1 \mapsto st^9 \\ x_2 \mapsto st^6 \\ z \mapsto \alpha t^5 - t^6 - \frac{2}{s}\alpha^2 t^7 + \frac{5}{s}\alpha t^8 + \dots \end{cases}$$

where $\mathbb{F} = \mathbb{Q}(s)[\alpha]$ with $\alpha^3 + s = 0$.

We want to check whether a differential form

$$\omega = p(x_1, x_2, z) \left(\frac{\partial g}{\partial z} \right)^{-1} \mathrm{d}\, x_1 \wedge \mathrm{d}\, x_2$$

is regular on a resolution.

Example Application: Adjoint Differential Forms

Under κ we get the following mappings:

$$\frac{\partial g}{\partial z} \mapsto 3s^2\alpha^2t^{22} - 3s^2t^{24} - 12s\alpha^2t^{25} + 42s\alpha t^{26} + \dots \quad \text{(order 22)}$$
$$\text{and } \mathrm{d}x_1 \wedge \mathrm{d}x_2 \mapsto -3st^{14}\,\mathrm{d}s \wedge \mathrm{d}t \quad \text{(order 14)}$$

Since $\mathrm{d}s \wedge \mathrm{d}t$ generates the module of regular differentials over $\mathbb{F}[[t]]$, we find that ω is regular in the local ring $\kappa^{-1}(\mathbb{F}[[t]])$ iff $\kappa(p)$ has order greater or equal than $22 - 14 = 8$.

Proceeding in the same way for the other embeddings, we eventually find that if p is a polynomial of degree less or equal than 3 then ω is regular on a resolution iff $p \in \langle x_1z^2 + x_1^2, x_1^2x_2, z^3 + x_1z, x_1x_2z, x_2z^2\rangle_{\mathbb{C}}$.

Implementation

We compute $\phi_0 : S_0 \to S$. Then we distinguish two cases:

1. **Divisors over Curves of the Discriminant:** We **change** the **ground field** to the function field of the corresponding curve and **compute univariate power series roots** right away.
2. **Divisors above Normal Crossings of the Discriminant:** We first **compute bivariate power series roots** and then transform each into a finite number of embeddings of the desired form. This step corresponds to the **resolution of a toric surface**.

Plus: Just linear algebra (and factorization)
Plus: Neat output
Minus: Requirements on field arithmetics ($\Rightarrow$ Magma)
Minus: Technically difficult (long code)

Conclusion

1. We implemented an analogue of the Newton-Puiseux algorithm for the surface case.
2. The implementation relies on an exact representation for power series.
3. First tests show good performance.
4. Thanks to Gavin Brown!