A DISCURSUS ON 21 AS A BOUND FOR RANKS OF ELLIPTIC CURVES OVER Q, AND SUNDRY RELATED TOPICS

1. Introduction

In the last paragraph of [32, §4], it was mentioned that a "forthcoming paper" of the author and A. Granville would give (heuristic) weight to the idea that (particularly if one ignores arithmetic issues) the number of curves of height up to X with rank r should be bounded by $X^{(21-r)/24+\epsilon}$ for $r \geq 1$. Some related ideas appeared in [34, §11], while herein we shall speak more directly of the general case of elliptic curves over $\mathbf{Q}$ (not just in a twist family). As can be surmised, much of this work is strongly influenced by conversations with A. Granville.

1.1. **Notation.** We shall find need to speculate on the number of curves of rank r with: real-period reciprocal $1/\Omega$ in a dyadic interval of size $1/\mathbf{\Omega}$; regulator in a dyadic interval of size $\mathbf{R}$; and (absolute) discriminant in a dyadic interval of size $\mathbf{\Delta}$. We denote this count by $\mathbf{N}(r, \mathbf{R}, \mathbf{\Omega}, \mathbf{\Delta})$ where notation can be dropped to indicate the counts without such restrictions.[1] We also use the notation $\mathbf{P}$ in place of $\mathbf{N}$, where this is the probability rather than a count, and bars over the arguments indicate which are unconditional.[2] The "$\approx$" symbol will be used in a very liberal manner, essentially meaning a log-asymptotic, and similarly with the "$\ll$" symbol. On the other hand, the "$\sim$" symbol indicates a variable in a dyadic interval, e.g., $a \sim A$ says $a \in [A, 2A]$. Finally, we often omit absolute values with the a, b coefficients of an elliptic curve, with its discriminant, or for the co-ordinates of a point. For instance, saying "$\log \Delta_E$" should not cause any confusion when $\Delta_E < 0$.

2. Points in ellipsoïds

Let E be an elliptic curve (over $\mathbf{Q}$) of rank r and regulator Reg_E. By counting lattice points in ellipsoïds, one finds [21, Theorem 20.4.2] that the number of points on E of canonical height less than H is asymptotically $(\gamma_r \cdot \#\mathrm{Tors}_E) \cdot H^{r/2}/\sqrt{\mathrm{Reg}_E}$ as $H \to \infty$, where $\#\mathrm{Tors}_E \leq 16$ by Mazur's theorem [25] and γ_r is the volume of the r-dimensional unit ball – we shall suppress these latter factors in the sequel. One expects this to be valid once H is sufficiently larger than the height of the largest generator (of a minimal generating set) of $E(\mathbf{Q})$. A conjecture of Lang [23, p. 92] predicts that canonical heights are $\gg \log \Delta_E$, and so the above asymptotic should be (at least) within a constant when $H \gg \mathrm{Reg}_E$. For most curves (those without skewed Mordell-Weil lattices), the switch-over should be around $H \approx \mathrm{Reg}_E^{1/r}$.

There is also the difference between canonical and naïve heights. From a result of Silverman [30] these differ by no more than $\mathrm{ht}(\Delta_E) + \mathrm{ht}(j_E)$, which by the ABC-conjecture [26] is $\ll \mathrm{ht}(\Delta_E) = \log \Delta_E \ll \mathrm{Reg}_E$. So for $H \gg \mathrm{Reg}_E$ we have

$$(1) \qquad \frac{H^{r/2}}{\sqrt{\mathrm{Reg}_E}} \ll \text{ number of points up to naïve height } H \text{ on a rank } r \text{ curve,}$$

where for now we ignore the implied constant, considering it more in §7 below.

[1] For instance, $\mathbf{N}(\mathbf{\Omega}, \mathbf{\Delta})$ would be the number of curves with reciprocal real-period of size $1/\mathbf{\Omega}$ and discriminant of size $\mathbf{\Delta}$, with no restrictions on either the regulator or rank.

[2] So $\mathbf{P}(\bar{2}, \mathbf{R}, \mathbf{\Omega}, \mathbf{\Delta})$ is the probability that a curve with regulator, real period, and discriminant of specified size has rank 2, while $\mathbf{P}(\bar{\mathbf{\Omega}}, \mathbf{\Delta})$ is the probability that a curve with discriminant of size $\mathbf{\Delta}$ has real period of size $\mathbf{\Omega}$.

3. Recollections of real periods

Section 3.3 of [32] derives a heuristic count for the number of elliptic curves, and also weights it by $\sqrt{\Omega_E}$ to get a rank 2 heuristic. We can use a similar technique to (partially) investigate the distribution of $1/\Omega_E$. As in the cited source, for simplicity our description is only for the positive discriminant case, but there are no real difficulties in analogising it for negative discriminant.

We are thus interested in

$$I_\kappa = \int\int (1/\Omega)^\kappa du_4\, du_6$$

where u_4, u_6 play the rôle of continuous variables for c_4, c_6 of an elliptic curve (we are approximating a lattice point sum by an integral). We are ranging over curves with discriminant of size $\mathbf{\Delta}$ so that $(u_4^3 - u_6^2) \sim 1728\mathbf{\Delta}$, with $1/\Omega$ given by $\frac{1}{\pi}\mathrm{agm}(\sqrt{e_1-e_2}, \sqrt{e_2-e_3})$ with $e_1 > e_2 > e_3$ roots of $4x^3-(u_4/12)x-(u_6/216) = 0$. Here κ is a real parameter, and we shall see that any $\kappa < 2$ gives a convergent integral, whereas we took $\kappa = -1/2$ in [32, §3.3].

The analysis of [32, §3.3] performs a co-ordinate transform to separate the discriminant, leaving us with

$$I_\kappa \approx \mathbf{\Delta}^{5/6+\kappa/12} \int_0^\infty A(\mu)^\kappa B(\mu)\, d\mu,$$

where with the reciprocal of the real period we have $A(\mu) = \mathrm{agm}\big(\sqrt{\lambda}, \sqrt{\lambda+\mu}\big)$ with $\lambda = (\sqrt{\mu^4+\mu} - \mu^2)/2\mu$ (or $\mu\lambda(\mu+\lambda) = 1/4$), while $B(\mu) = 1/\sqrt{\mu^4+\mu}$ comes from the Jacobian of the transformation.

We thus see that $B(\mu) \to 1/\mu^2$ as $\mu \to \infty$ and $B(\mu) \to 1/\sqrt{\mu}$ as $\mu \to 0$, while $A(\mu) \approx \mu^{1/2}/\log\mu$ as $\mu \to \infty$ and $A(\mu) \to 1/\mu^{1/4}\sqrt{2}$ as $\mu \to 0$. This shows that the μ-integral converges for any $\kappa < 2$. We compare I_2 with (the convergent) I_κ for $\kappa = 2 - \frac{1}{\log\mathbf{\Delta}}$, which then induces an extra factor of $\log\mathbf{\Delta}$ with $\mu \to 0$.

Our heuristic[3] is then that the sum of $1/\Omega_E^2$ over curves E with discriminant of size $\mathbf{\Delta}$ is $\ll \mathbf{\Delta}\log\mathbf{\Delta}$, so that (suppressing the log) we have

$$\mathbf{N}(\mathbf{\Omega}, \mathbf{\Delta}) \ll \mathbf{\Delta}\mathbf{\Omega}^2 \quad\text{or}\quad \mathbf{P}(\bar{\mathbf{\Omega}}, \mathbf{\Delta}) \ll \mathbf{\Delta}^{1/6}\mathbf{\Omega}^2, \tag{2}$$

the latter using the $\kappa = 0$ version of the above, namely that we expect $\mathbf{N}(\mathbf{\Delta}) \approx \mathbf{\Delta}^{5/6}$ curves with discriminant of size $\mathbf{\Delta}$. Furthermore, we have $1/\Omega_E \gg \Delta_E^{1/12}$ as noted in [32, §6.2], so that $\mathbf{P}(\bar{\mathbf{\Omega}}, \mathbf{\Delta}) \approx 1$ when $1/\mathbf{\Omega} \approx \mathbf{\Delta}^{1/12}$. Stated differently, the typical size of the reciprocal of the real period is $\Delta_E^{1/12}$, a fact we shall use frequently below.

4. Heuristics with regulators

The next task will be to transfer a couple of estimates that are related to those in [32, §3] to consider the effect of a regulator. This is related to the real period and L-values by the Birch–Swinnerton-Dyer conjecture [2]. Although the L-value distribution does not seem to be correctly predicted by random matrix theory (if we have interpreted the latter properly), it is still in our minds when we later write down bounds for such distributions. The reader who prefers to skip this analysis

[3] Mostly we replaced lattice-point sums by areal integrals. Making this rigourous implies (for instance) the ABC conjecture, which is (roughly) equivalent (see [16]) to $1/\Omega_E \ll \Delta_E^{1/2}$.

can safely simply take $1/\Omega_E \approx \Delta_E^{1/12}$ and $\mathrm{Reg}_E \approx 1/\Omega_E$ in the succeeding sections, as the analysis here shows these are the dominant regions for our cases of interest.

An analysis as in [32, §3] gives an expectation of $\mathbf{N}(0,\boldsymbol{\Delta}) \approx \mathbf{N}(1,\boldsymbol{\Delta}) \approx \boldsymbol{\Delta}^{5/6}$ and $\mathbf{N}(2,\boldsymbol{\Delta}) \approx \boldsymbol{\Delta}^{19/24}$, the latter up to a $\boldsymbol{\Delta}^{o(1)}$ factor that we shall suppress throughout this paper (though we say more about it in §7 below).

Let us then try to consider $\mathbf{N}(1,\mathbf{R},\boldsymbol{\Omega},\boldsymbol{\Delta})$. Throughout this paragraph we consider E to have rank 1. We follow [32, (1)] at least at the rough level (the other effects should either wash out or be negligible $\boldsymbol{\Delta}^{o(1)}$ contributions), and from random matrix theory predict

$$\mathbf{Prob}[L_E'(1) \le t] \approx t^{3/2} \quad \text{as } t \to 0.$$

As the Tamagawa number (and other) contributions to the BSD-formula should be $\boldsymbol{\Delta}^{o(1)}$, assuming Ш$_E$ is not typically large (see [7]) we restate this roughly as

$$\mathbf{Prob}[\mathrm{Reg}_E \le t/\Omega_E] \approx t^{3/2} \quad \text{as } t \to 0.$$

Using $\mathbf{N}(1,\boldsymbol{\Delta}) \approx \boldsymbol{\Delta}^{5/6}$ we find that $\mathbf{N}(1,\mathbf{R},\boldsymbol{\Omega},\boldsymbol{\Delta}) = \mathbf{N}(1,\boldsymbol{\Delta}) \cdot \mathbf{P}(1,\bar{\mathbf{R}},\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta})$ is

$$\approx \boldsymbol{\Delta}^{5/6} \cdot \mathbf{P}(1,\bar{\mathbf{R}},\boldsymbol{\Omega},\boldsymbol{\Delta}) \cdot \mathbf{P}(1,\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}) \approx \boldsymbol{\Delta}^{5/6} \cdot \mathbf{R}^{3/2}\boldsymbol{\Omega}^{3/2} \cdot \mathbf{P}(1,\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}).$$

4.1. Yet this looks wrong for curves with small regulator, say $\mathbf{R} \approx \log \boldsymbol{\Delta}$, for it predicts $\mathbf{N}(1,\mathbf{R},\boldsymbol{\Delta}) \ll \boldsymbol{\Delta}^{5/6} \cdot \boldsymbol{\Delta}^{(-1/12)(3/2)} = \boldsymbol{\Delta}^{17/24}$ (recall $1/\boldsymbol{\Omega} \gg \boldsymbol{\Delta}^{1/12}$), while we can[4] construct $\boldsymbol{\Delta}^{3/4}$ such curves via $(x,y,a) \sim (\boldsymbol{\Delta}^{1/6},\boldsymbol{\Delta}^{1/4},\boldsymbol{\Delta}^{1/3})$ and solving for b in $y^2 = x^3 + ax + b$. In fact, we shall use these points of small height later when determining how to calibrate the final rank heuristic.

However, we still might guess a bound $\mathbf{Prob}[L_E'(1) \le t] \ll t^{1/2}$ as $t \to 0$, and similarly for higher derivatives (rather than increasing the t-exponent with the order of the derivative,[5] as per random matrix theory). The above analysis then finds that $\mathbf{P}(r,\bar{\mathbf{R}},\boldsymbol{\Omega}) \ll \mathbf{R}^{1/2}\boldsymbol{\Omega}^{1/2}$, and so

$$\mathbf{P}(\bar{r},\bar{\mathbf{R}},\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}) = \mathbf{P}(r,\bar{\mathbf{R}},\boldsymbol{\Omega},\boldsymbol{\Delta}) \cdot \mathbf{P}(\bar{r},\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}) \ll (\mathbf{R}\boldsymbol{\Omega})^{1/2} \cdot \mathbf{P}(\bar{r},\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}) \tag{3}$$

This is perhaps not optimal, but does look reasonably viable as a heuristic.

Furthermore, we should also indicate the places where $\mathbf{P}(\bar{r},\bar{\mathbf{R}},\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}) \approx \mathbf{P}(\bar{r},\boldsymbol{\Delta})$. From (3) we see $\mathbf{R} \approx 1/\boldsymbol{\Omega}$ maximises as a function of $\mathbf{R}$. On the other hand, (by dyadicity) there must be some $\boldsymbol{\Omega}$ with $\mathbf{P}(\bar{r},\boldsymbol{\Delta}) \approx \mathbf{P}(\bar{r},\boldsymbol{\Omega},\boldsymbol{\Delta}) \cdot \mathbf{P}(\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta})$. As (2) bounds the latter factor rather significantly when $1/\boldsymbol{\Omega} \not\approx \boldsymbol{\Delta}^{1/12}$, we would need for $\mathbf{P}(\bar{r},\boldsymbol{\Omega},\boldsymbol{\Delta})$ to be quite larger than $\mathbf{P}(\bar{r},\boldsymbol{\Delta})$ for such a $\boldsymbol{\Omega}$ to dominate, which seems unlikely (indeed, with $\mathbf{P}(\bar{2},\boldsymbol{\Omega}) \approx \sqrt{\boldsymbol{\Omega}}$ the effect is the opposite). So we expect

$$\mathbf{P}(\bar{r},\bar{\mathbf{R}},\bar{\boldsymbol{\Omega}},\boldsymbol{\Delta}) \approx \mathbf{P}(\bar{r},\boldsymbol{\Delta}) \quad \text{when } \mathbf{R} \approx 1/\boldsymbol{\Omega} \approx \boldsymbol{\Delta}^{1/12}. \tag{4}$$

5. Granville's heuristic

The idea of Granville's heuristic is that one can guesstimate an upper bound on the number of (integral) points (up to a specific height) on a "covering" variety that contains all the points (up to that height) on the elliptic curves in a given discriminant range. This upper bound is then compared to the lower bound that comes from counting lattice points in ellipsoïds (§2), though it is only in succeeding sections that we shall consider how to collate the bounds.

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[4]One can ask whether these curves actually do typically have rank 1; we cannot prove this is the case, but it seems likely that (at the very least) a positive proportion should be rank 1.

[5]In the twist case, the 3/2 exponent for the $L_E'(1)$-distribution fits the data of [33, §3.2].

Let us consider the "family" of all elliptic curves over $\mathbf{Q}$, which we shall order in the "big box" enumeration (though see Footnote 6). We can write a elliptic curve projectively as $y^2z = x^3 + axz^2 + bz^3$ and put $z = \tilde{z}^3$ and $x = \tilde{x}\tilde{z}$, so that $y^2\tilde{z}^3 = \tilde{x}^3\tilde{z}^3 + a\tilde{x}\tilde{z}^7 + b\tilde{z}^9$ or

$$V : Y^2 = X^3 + aXZ^4 + bZ^6.$$

We wish to count (or at least bound) the number $G_{\mathbf{\Delta}}(T)$ of integral (a, b, X, Y, Z) points on V with these variables in dyadic ranges. We want the curve to have discriminant of size $\mathbf{\Delta}$, and will approximate[6] this situation by $\max(a^3, b^2) \sim \mathbf{\Delta}$. We furthermore want the point to have height of size T, so that $\max(X, Z^2) \sim T$.

5.1. **A probabilistic model.** One crude idea is to model the right hand of the equation for V by it being an integer of size[7] $S = \max(X^3, aXZ^4, bZ^6)$, and thus has a probability $1/\sqrt{S}$ of being square. We have $\mathbf{\Delta}^{5/6}$ choices for (a, b), while the number of (X, Z) choices depends on the ranges in the maximum defining S. For instance, when X^3 dominates the S-maximum, we have $X \approx T$ and the number of possible Z is $\min(\sqrt{T}/a^{1/4}, \sqrt{T}/b^{1/6}) = \sqrt{T}/\max(a^{1/4}, b^{1/6}) \sim \sqrt{T}/\mathbf{\Delta}^{1/12}$, and so the probability estimate suggests that number of such integral points on the variety V is approximately

$$\frac{\mathbf{\Delta}^{5/6} \cdot T \cdot \sqrt{T}/\mathbf{\Delta}^{1/12}}{\sqrt{T^3}} = \mathbf{\Delta}^{3/4}.$$

We consider the other cases (a bit tediously) of S-dominance in the next two subsubsections, where again $\mathbf{\Delta}^{3/4}$ will be the principal contribution.

5.1.1. *When bZ^6 dominates.* The second case has bZ^6 dominating the S-maximum, so we have that $X \ll \min(T, Z^2b/a, Z^2b^{1/3})$. First consider the $Z^2 \approx T$ subcase, when the probability of Y being square is $1/\sqrt{b}T^{3/2}$. The number of Z-choices is thus $\sqrt{T}$ while the number of X-choices is bounded as above, and so for a given (a, b) pair its probability of appearing in a V-point (with the relevant height restrictions) is $\min(1, b/a)/\sqrt{b}$, We then sum over the (a, b)-ranges, recalling $\max(a^3, b^2) \approx \mathbf{\Delta}$. When $b^2 \approx \mathbf{\Delta}$ the probability is just $1/\sqrt{b} \approx \mathbf{\Delta}^{1/4}$, yielding $\ll \mathbf{\Delta}^{5/6-1/4} = \mathbf{\Delta}^{7/12}$ points on V. Indeed, the same analysis holds when $|b| \geq |a| \approx \mathbf{\Delta}^{1/3}$, for which we find $\ll \mathbf{\Delta}^{5/6}/\sqrt{\mathbf{\Delta}^{1/3}} = \mathbf{\Delta}^{2/3}$ points on V. Finally, when $|a| \approx \mathbf{\Delta}^{1/3}$ and $b \leq |a|$, there are at most $a^2 \approx \mathbf{\Delta}^{2/3}$ pairs to count, yielding the same bound on V-points.

The second subcase here is when $X \approx T$, where the probability that Y is a square is $1/\sqrt{b}Z^3$. The prediction of the total number of V-points will thus be $T/\sqrt{b}Z^3$ summed (or effectively integrated) over the relevant (a, b, Z) regions. We see that $Z^2 \gg \max(Ta/b, T/b^{1/3})$ for bZ^6 to be dominant. By exercising the sum over Z we are left with $\min(b/a, b^{1/3})/\sqrt{b} = \min(\sqrt{b}/a, 1/b^{1/6})$. When $|a| \approx \mathbf{\Delta}^{1/3}$ the minimum is less than $\sqrt{\mathbf{\Delta}^{1/2}}/\mathbf{\Delta}^{1/3}$, saving $\mathbf{\Delta}^{1/12}$ for a bound of $\mathbf{\Delta}^{3/4}$ on the count of V-points. When $|b| \approx \mathbf{\Delta}^{1/2}$ similarly $1/b^{1/6}$ saves $\mathbf{\Delta}^{1/12}$ with the same result.

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[6] It should be reasonable to (in the heuristic sense of areal integrals approximating lattice point sums) consider $4a^3 + 27b^2 \approx \Delta$ instead, but the argument does not add anything to the heuristic and would tend to obscure the point.

[7] This could be "refined" by separately considering regions where there is some cancellation between these three quantities, but my calculations show that this does not matter in the end, and again would obscure the issue. E.g., the number of (a, b, X, Z) with $|aXZ^4 - bZ^6| \approx T^{3-\delta}$ is of size $T^{3/2-\delta}$ (up to $\mathbf{\Delta}$ adjustments), which then saves $1/T^{\delta/2}$ in the subsequent analysis.

5.1.2. *When aXZ^4 dominates.* Finally we consider the case when aXZ^4 dominates the S-maximum. The power-mean inequality implies $(X^3)^{1/3}(bZ^6)^{2/3} \ll aXZ^4$, so that $b^{2/3} \ll a$ and we must have $a \approx \mathbf{\Delta}^{1/3}$. Again it is profitable to first consider the case when $X \approx T$. Then the probability of Y being a square is $1/\sqrt{aTZ^4}$, and summing over Z with use of $T^2 \ll aZ^4$, we find that relevant expression to be summed over (a, b) is $1/a^{1/4}$. This saves $1/\mathbf{\Delta}^{1/12}$ and gives an upper bound of $\mathbf{\Delta}^{3/4}$ in the V-count. Similarly, in the case where $Z^2 \approx T$ the probability of Y being square is $1/\sqrt{aXT^2}$, and summing over $X \ll T$ finds $1/\sqrt{a}$ to be summed over the (a, b) region. As before this saves $1/\mathbf{\Delta}^{1/6}$, yielding an acceptable $\mathbf{\Delta}^{2/3}$ bound for the count of V-points.

5.2. **Conclusion.** We conclude[8] that there should be about $\mathbf{\Delta}^{3/4}$ points on V that satisfy both $\max|a^3, b^2| \approx \mathbf{\Delta}$ and $\max|X, Z^2| \approx T$. Upon having this estimate $G_{\mathbf{\Delta}}(T) \approx \mathbf{\Delta}^{3/4}$ (which is, perhaps curiously, independent of T), we then sum it dyadically over T up to a bound e^H, which gives $H\mathbf{\Delta}^{3/4}$. So we can write Granville's heuristic as

$$\#\{\text{points up to naïve height } H \text{ on all curves with } \Delta_E \approx \mathbf{\Delta}\} \ll H\mathbf{\Delta}^{3/4}, \tag{5}$$

and the task becomes to determine/predict in what H-range this might be valid. Furthermore, we might note that Granville's heuristic could additionally predict that the two sides are approximately equal (in the sense used in other places of this paper) in such a range.

6. Combining estimates

We now wish to compare the estimate (1) from lattice point counting to the estimate (5) of Granville. From this we might obtain (for each r)

$$\mathbf{\Delta}^{5/6} \sum_{(\mathbf{R})} \mathbf{P}(\bar{r}, \bar{\mathbf{R}}, \mathbf{\Delta}) \cdot \frac{H^{r/2}}{\sqrt{\mathbf{R}}} \ll H\mathbf{\Delta}^{3/4}, \tag{6}$$

where the left side is a sum over the dyadic $\mathbf{R}$-intervals. We can string out the summation further, introducing a dyadic $\mathbf{\Omega}$-sum to get

$$\sum_{(\mathbf{R})} \sum_{(\mathbf{\Omega})} \frac{\mathbf{P}(\bar{r}, \bar{\mathbf{R}}, \bar{\mathbf{\Omega}}, \mathbf{\Delta})}{\sqrt{\mathbf{R}}} \ll H^{1-r/2}\mathbf{\Delta}^{-1/12},$$

where by (4) the $\mathbf{R} \approx 1/\mathbf{\Omega} \approx \mathbf{\Delta}^{1/12}$ term contributes $\mathbf{P}(\bar{r}, \mathbf{\Delta})\sqrt{\mathbf{\Omega}} \approx \mathbf{P}(\bar{r}, \mathbf{\Delta})/\mathbf{\Delta}^{1/24}$ to the double-sum on the left. On the other hand, by (3) this double-sum on the left is bounded above (up to logs from the $\mathbf{R}$-dyadicity) by

$$\sum_{(\mathbf{R})} \sum_{(\mathbf{\Omega})} \mathbf{P}(\bar{r}, \bar{\mathbf{\Omega}}, \mathbf{\Delta}) \cdot \mathbf{\Omega}^{1/2} \ll \frac{1}{\mathbf{\Delta}^{1/24}} \sum_{(\mathbf{\Omega})} \mathbf{P}(\bar{r}, \bar{\mathbf{\Omega}}, \mathbf{\Delta}) = \frac{\mathbf{P}(\bar{r}, \mathbf{\Delta})}{\mathbf{\Delta}^{1/24}},$$

confirming our claim of the dominance of the $\mathbf{R} \approx 1/\mathbf{\Omega} \approx \mathbf{\Delta}^{1/12}$ term.

Putting this back into (6), we find that

$$\mathbf{P}(\bar{r}, \mathbf{\Delta}) \ll H^{1-r/2}\mathbf{\Delta}^{3/4-5/6+1/24} = H^{1-r/2}\mathbf{\Delta}^{-1/24},$$

[8] One can try to bolster such analysis by the use of congruential information as in [34, §9], therein relying on a sort of equidistribution in exponential ranges first considered by Hooley [20] regarding the Pell(ian) equation, but such additions do not change the result. In the quadratic twist case of [34, §9], the analysis of congruences did change the result in the case where the curve has 2-torsion, but here we know [9, 17] that such curves are sparse.

and as can be seen, when $r = 2$ the H-dependence drops out and we recover[9] the (crude) bound of $\mathbf{N}(2, \mathbf{\Delta}) \ll \mathbf{\Delta}^{19/24}$. For $r > 2$ we obtain a better bound by taking H as large as possible.[10] For instance, if we could take $H = \mathbf{\Delta}^{1/12}$ we would then have $\mathbf{P}(\bar{r}, \mathbf{\Delta}) \ll \mathbf{\Delta}^{(1-r)/24}$ which says $\mathbf{N}(r, \mathbf{\Delta}) \ll \mathbf{\Delta}^{(21-r)/24}$ and leads[12] to $r \leq 21$. However, we can also consider this comparison when $r = 1$. Indeed the analysis of this will aid our guess for what size we can take H to be.[13]

6.1. **Calibration.** The idea (which admittedly has an *ad hoc* nature to it) is that we can take H up to the point where the rank 1 curves no longer give a substantial proportion of the points that are allowed on the right side of (5), and no further.

Firstly, we can note that as $H \to \infty$, the number of points (on V) from curves of rank 1 and discriminant of size $\mathbf{\Delta}$ should be approximately[14] $\sqrt{H}\mathbf{\Delta}^{5/6-1/24}$, which is eventually smaller than Granville's $H\mathbf{\Delta}^{3/4}$. So there must be some "transition" point when the rank 1 curves start producing less points than Granville allows.

Now for small H, there is not really a discrepancy (in rank 1) when comparing the ellipsoïd count $\sqrt{H}\mathbf{\Delta}^{19/24}$ to the Granville bound $H\mathbf{\Delta}^{3/4}$ — namely, we can only apply the ellipsoïd counting when $H \gg \mathrm{Reg}_E$, and so the proposed estimate needs reworking. Indeed, from the examples[15] of small points in §4.1, we might be led to think that there are $H\mathbf{\Delta}^{3/4}$ curves of rank 1 with discriminant of size $\mathbf{\Delta}$ and regulator of size e^H, at least until H becomes large enough for this to exhaust the totality of the $\mathbf{\Delta}^{5/6}$ rank 1 curves with discriminant of size $\mathbf{\Delta}$. And indeed, this cross-over point is precisely when $H \approx \mathbf{\Delta}^{1/12}$.

7. Error terms and the Elkies rank 28 curve

We now give some indication of the error terms for the quantities we estimated above, and how these might allow larger ranks than 21 to occur for "small" discriminants (or conductors). There are two types of such error terms, those that depend on the rank r, and those that do not. A prototypical version of the latter is the product of local Tamagawa numbers, which can be (crudely) bounded similar to a divisor function, and thus (see [36, §4]) is of size no more than $N^{c/\log\log N}$.

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[9] The reader may inquire whether we have fulfilled the prospectus of [32, §4.5], namely to produce a "different" method to produce $X^{19/24}$. Indeed, some of the above uses estimates on L-values (and so might be thought of as like the argument in [32, §3]), but that really obfuscates the picture. We are, in fact, mostly only using the condition that $\mathbf{\Omega} \approx 1/\mathbf{\Delta}^{1/12}$ dominates the analysis, which does not depend upon L-values (they only serve to give more confidence that this condition might be correct). Thus the argument here, that $\mathbf{\Delta}^{19/24}$ comes from the co-incidence of $H^{r/2}/\sqrt{\mathrm{Reg}}$ with a point-count bound $H\mathbf{\Delta}^{3/4}$, should be seen as different from both the L-value analysis of [32, §3], and the Ш-is-a-random-square analysis that is briefly mentioned in [32, §4.5].

[10] Indeed, one can immediately note that we get that $\mathbf{N}(3, \mathbf{\Delta})$ is significantly less than $\mathbf{\Delta}^{19/24}$ unless one can only take H to be an insignificant function of $\mathbf{\Delta}$. Combined with the suspicion[11] that $\mathbf{N}(2, \mathbf{\Delta}) \approx \mathbf{\Delta}^{19/24}$, one would then have the number of rank 3 curves is significantly less than the number of rank 2 curves, which for some reason is occasionally disputed.

[11] The method given here does not really support such a (crude) asymptotic, only an upper bound, but the other two methods of two footnotes previous both lend credence to this prediction.

[12] Generally, if curves with $\mathbf{R} \approx \mathbf{\Delta}^{\kappa}$ dominate (for rank r) the left side of (6), taking $H \approx \mathbf{\Delta}^{\eta}$ in the calibration gives $(r/2-1)\eta \leq 3/4+\kappa/2$ (where BSD with ABC+PL implies $\kappa \leq 1/2+1/4$).

[13] This is (partially) distinct to the GPPVW heuristic [15], where $r = 2$ is used for calibration.

[14] Here the $1/24$ comes from the typical size of the square root of the reciprocal of the real period, and indeed this real-period reciprocal is itself the typical size of the regulator.

[15] The comments here might propose $\mathbf{P}(1, \bar{\mathbf{R}}, \mathbf{\Omega}, \mathbf{\Delta}) \approx \mathbf{P}(1, \mathbf{\Delta}) \cdot (\mathbf{R}\mathbf{\Omega}) \approx \mathbf{R}\mathbf{\Omega}$ in place of (3).

Examples of the first type include L-values, (effects of) differences between naïve and canonical heights, and our suppression of the volume γ_r of the r-dimensional unit ball. A result of Brumer gives us $r \lesssim \frac{\log N}{2\log\log N}$ under GRH, but in reality, if we can take H to be any positive power of $\mathbf{\Delta}$ in the calibration (§6.1), then the rank will be bounded and the asymptotic analysis will be superfluous.

The (asymptotically) most significant correction is $\pi^{r/2}/\Gamma(1+r/2)$ from the volume of the r-dimensional unit ball, which will tend to $1/N^{1/4}$ as $r \to \frac{\log N}{2\log\log N}$ as $N \to \infty$. At first glance one might think would this would modify the Granville-versus-ellipsoïd comparison to be $H^{r/2}/\sqrt{R_E} \ll H\mathbf{\Delta}^{3/4+1/4}$, but as just noted, one would still obtain a prediction of an upper bound of (say) $r \le 27$ with $H \approx \mathbf{\Delta}^{1/12}$, and then with this bound in hand the effect of the γ_r is negligible asymptotically (that is, just a constant) as $\mathbf{\Delta} \to \infty$. Similarly, for the leading Taylor series term, since $L_E(s)$ acts like N^s at infinity, we expect that the rth derivative is bounded by $(\log N)^r$, and then the Brumer bound implies that the effect[16] of $L_E^{(r)}(1)/r!$ as $N \to \infty$ is no more than $N^{f(N)}$ with $f(N) \approx \frac{\log\log\log N}{2\log\log N}$. Finally, the adjustment for the difference between canonical and naïve heights is also not more than c^r for some constant c (and is negligible for points for points whose height is not $\ll \log\Delta$).

In short, while we might expect the typical regulator is $\mathrm{Reg}_E \approx 1/\Omega_E \approx \Delta_E^{1/12}$ asymptotically, the right side can have another term of size $(\log N_E)^r/r!$; meanwhile, the count of lattice points should have an extra factor of γ_r included.

7.1. **The Elkies curve.** This curve [12] of rank ≥ 28 is defined by the a-invariants

$$[1, -1, 1, -20067762415575526585033208209338542750930230312178956502,$$
$$3448161179503055646703298569039072037485594435931918036126600829629193944873224342 9].$$

This is semistable at all bad primes except 3, where it has local conductor 3^2. The discriminant/conductor ratio is 103651294268844241920000 0, or approximately 15% of the size of the conductor, while the Tamagawa product is 172800. The reciprocal of the real period is fairly well approximated by $\Delta_E^{1/12}$, off by a factor of about 4.41. The canonical heights of the points listed by Elkies are all about 48, while their naïve heights are typically closer to 63. This might involve a discrepancy of $\approx (4/3)^{28} \approx 3000$ in estimates with points of (logarithmically) small height. Perhaps most crucially, the regulator is $\approx 3.8573 \cdot 10^{34}$, about $27 \cdot 10^{20}$ times the reciprocal of the real period, so greatly throwing off the previous expectations.

More specifically, we find that $L_E^{(r)}(1)/r! \approx 7 \cdot 10^{40}$ for this curve, notably large compared to $\Delta_E^{1/12} \approx 6.25 \cdot 10^{13}$, which implies that we are not yet near the asymptotic regime. Indeed, even the most optimistic analysis with error terms should lead one to predict that curves of rank ≥ 22 are still permitted until a range much beyond what has currently been considered, for instance until $\frac{1}{\log\log N_E} \ge 1/12$, with most likely this right side needing to be reduced by another nonnegligible factor.

On the other hand, for the comparison[17] of $H^{r/2}/\sqrt{R_E}$ versus $H\Delta_E^{3/4}$, upon taking the suggested $H = \Delta_E^{1/12} \approx 6.25 \cdot 10^{13}$ we find that $H^{r/2}/\sqrt{R_E} \approx 7 \cdot 10^{175}$ which is much larger than $H\Delta_E^{3/4} \approx 9 \cdot 10^{137}$, and this perhaps casts some doubt on the Granville estimate already. However, taking instead $H = \Delta_E^{1/12}/\log\Delta_E$ malleates the estimates into reasonably close agreement (particularly with γ_{28} included).[18]

[16]The bound with N_E can be translated to Δ_E by the ABC (or Szpiro's) conjecture [26, §2].

[17]The volume γ_{28} of the unit 28-sphere is $\approx 10^{-4}$, so the correction it induces is rather minor.

[18]Yet in this regard, the same could be argued for $H \approx \exp(\sqrt{\log\Delta_E})\log\Delta_E$.

8. Note on Granville's heursitic in the quadratic twist case

In §13.3 of [34], we cite Rubinstein as presenting a challenge of taking a curve in Cremona's database without 2-torsion, and finding a quadratic twist of rank 6.

This has since been answered by Daniels and Goodwillie [6, Table 4] already for 11a, noting that the -203145767th twist has rank 6, and similarly the twist of 550k by 4817182.

9. Higher degree or weight

Viewing elliptic curves over $\mathbf{Q}$ as avatars for degree 2 L-functions of motivic weight 1, one can then ask similar questions about (analytic) ranks of other odd[19] weight motives. This then is an excuse for me to record (all in one place) about as much as I know/suspect here.

9.1. Degree 2.

9.1.1. *Motivic weight 1.* Although it is not directly related to the ideas here, I would be remiss not to point out that there is a modular form

$$f = q - iq^2 - 2q^3 - q^4 - 3q^5 + 2iq^6 + 3iq^7 + iq^8 + q^9 + 3iq^{10} - 3iq^{11} + \dots$$

over $\Gamma_1(122)$ that is defined over $\mathbf{Q}(\sqrt{-1})$ and has $L_f(1) = 0$ even though the root number (a root of $61x^4 - 22x^2 + 61$) does not force this.[20] This was noted in September 2000 by W. A. Stein, using his Magma modular forms implementation.

It seems this is the first example with quadratic character; earlier examples exist at levels 61, 63, 80, 85, 91, 101, 104, 105, 111, 112, 114, and 117 (three).

There are also the calcuations of Brumer [4], which give examples of analytic rank 2 for higher-dimensional quotients of $J_0(N)$, that is, those where the Fourier coefficients of the modular form live in a (totally real) number field of higher degree.

9.1.2. *Motivic weight 3.* These correspond to modular newforms of weight 4. Here we might expect analytic rank 2 to be rather common. For instance, if we fix a modular newform f with rational coefficients and consider its quadratic twists, the Waldspurger formula [31] (see also [29] and [22]) gives an associated modular form of weight 5/2 whose dth coefficient is related to the central L-value of the dth twist of f, at least for d which satisfy an arithmetical relation with the level N of f (for instance, d should be a square modulo $4N$). As we expect that this weight 5/2 form has dth coefficient of size $d^{3/4}$, this then gives a prediction that in any quadratic twist family, there should be approximately $D^{1/4}$ twists up to D which have analytic rank 2 (or higher). Indeed, this is reasonably well borne out by the data of Rubinstein [28].

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[19] Typically even weight motives do not have vanishing central L-value (it is not a special value in the sense of Deligne [8]), though the construction of Armitage [1] yields motivic weight 0 L-functions with odd parity. Friedlander [14] gives an explicit example, and the "smallest" one appears to be an Artin representation of conductor $2^8 3^2$, for the field $x^8+12x^6+36x^4+36x^2+9$, though perhaps more interesting is the $\mathrm{SL}_2(\mathbf{F_3})$ example of conductor 163^2 associated to the field $x^8 - x^7 + x^6 - 4x^5 + 5x^4 - 8x^3 + 4x^2 - 8x + 16$.

[20] In [27, Exercise 5.5] Rohrlich gives a different example, a twist by $-118 - 18\sqrt{-7}$ of 49a.

We can also list small levels for which there is a rational newform of analytic rank 2. Up to 1000, the list is

$$127, 159, 365, 369, 453, 465, 477, 567, 581, 639, 657, 681,$$
$$684, 781, 832, 848, 855, 864, 885, 892, 918, 945, 957, 969,$$

and we entrust the reader can determine which form at the given level has the extra vanishing of its L-function. Many of these appear in [10], either in Table 1 or the final paragraph of §7.4 therein. We computed (using Magma) every space of modular forms of weight 4 and level up through 2900, and found no examples[21] of analytic rank 3 for degree 2 motivic weight 3 L-functions (see [19, V.4] and [37] for the arithmetic context).

Similar to above note concerning non-selfdual data, there are weight 4 modular form of levels 68, 77, and 99, with vanishing L-value, with this last one having quadratic character (others with quadratic characters exist at levels 106, 120, 128, and 200; I did not check higher weights for this).

9.1.3. *Motivic weight 5.* Here is a list of the levels up to 1000 that have a rational newform with analytic rank 2:

$$95, 116, 122, 260, 308, 359, 371, 400, 470, 527, 539, 566, 700, 770, 832, 880, 980.$$

Again the first few examples appeared in [10], and we entrust the reader can determine which form[22] at the relevant level is the one whose L-value has extra vanishing. Moreover, Table 5 of [32] lists some twist data[23] for symmetric powers of Grössencharacters attached to CM elliptic curves, yielding 16 more examples, the one of largest conductor being the 89320th twist of ψ^5 for ψ attached to 256b.

9.1.4. *Motivic weight 7.* Up through level 1000, there are four rational newforms with analytic rank 2, at levels 425, 585, 825, and 957. The first is the 5th quadratic twist of the level 17 newform, the second is the -3rd twist of the rational level 195 newform, and the third is the 5th quadratic twist of the level 33 newform. Again Table 5 of [32] has 3 additional examples, the most spectacular being the 27365th twist of ψ^7 for ψ attached to 121b.[24] Furthermore, Footnote 25 of [32] recalls[25] the computations in the early 80s of N. M. Stephens, which found that ψ^7 for ψ attached to the elliptic curve $[0, 0, 0, 127, 0]$ has analytic rank 2.

[21] All "analytic ranks" that we report are experimental observations, though in some cases one can give an arithmetical criterion that shows the L-value is truly zero.

[22] Perhaps the most notable one is the level 832 example, which is the -8th quadratic twist of the level 52 newform with $a_3 = -5$. Note that the (motivic) weight 3 example at this level is also a quadratic twist (of a level 26 newform). The level 400 example is also a twist, of the level 50 newform with $(a_2, a_3) = (4, 11)$, as is the level 880 example, of the rational newform of level 440.

[23] The use of the Waldspurger formula yields a weight 7/2 form with dth coefficient of size $d^{5/4}$, so it would be somewhat surprising if there were many vanishings in the twist family of a fixed modular form. Indeed even just one vanishing twist with $|d| \geq 10^4$ is rather unexpected, while 8 such examples (across 10 CM curves tested) are listed in [32, Table 5].

[24] The Table erroneously calls this "121a", and with regards to the previous footnote, we now have a weight 9/2 form from the Waldspurger formula, with dth coefficient of size $d^{7/4}$.

[25] Greenberg's paper [18, p. 243] lists two examples of ψ^3 having extra vanishing, but when we visited him in November 2006, he was able to show us ancient computer printouts from Stephens that included this ψ^7 example, whose conductor is $2^5 \cdot 127^2 = 516128$.

9.1.5. *Higher weights.* It seems to me that sufficiently higher (motivic) heights, say 16, are quite unlikely to yield examples of analytic rank 2. However, I really cannot guess what the crossovers from "infinitely many" to "finitely many" to "none" are. Probably weight 5 has infinitely many, while I think I would guess weight 7 does also, while weight 11 might already have none. I'm not sure what I would guess for weight 9, where there no examples up through level 750.

In analogy with the "generic" guess[26] that a weight 2 rational newform of level N has a $1/N^{1/24}$ chance of having analytic rank 2, one might propose that a weight $2k$ newform defined over a field of degree d has a $1/N^{d(2k-1)/24}$ chance of having analytic rank 2. One would still need to know, however, an estimate for the number of such "generic" (non-twist) forms at a given level, which seems a difficult problem.

9.2. **Degree 4.** Next we turn to the case of (primitive) degree 4 L-functions. The examples of (motivic) weight 1 here will be Jacobians of genus 2 curves or elliptic curves over quadratic fields. Again we expect that these can have quite high ranks.[27]

9.2.1. *Motivic weight 3, symmetric cubes.* The first examples to be computed here seem to be symmetric cube L-functions of elliptic curves [5]. Indeed, Buhler, Schoen, and Top already found (Table 9.3) that 39a has a symmetric cube with analytic rank 2, and listed two examples (2379b and 31605ba) of analytic rank 4. Note that these conductors are distinctly less than their counterparts for elliptic curves, though one must (typically) cube the conductor with Sym^3. The computations of [24] found 14 more examples (see Table 7 therein) of analytic rank 4 for elliptic curves with conductor up to 130000 (the limit of Cremona's tables at the time), but none of analytic rank 5.

Perhaps more interesting in Table 6 of [24], which notes that there were only 16782 examples of analytic rank 2 for the symmetric cube in this conductor range, compared to 61787 for the elliptic curves themselves, while for analytic rank 3 it was about even (908 vs 905), and for analytic rank 4 the symmetric cube produced 16 examples, and the elliptic curves themselves 0.

The data in [32, §6.6] also mention a total of 6 cases of analytic rank 3 when looking at quadratic twists of the symmetric cube of 11a, 14a, or 15a, and a total of 229 cases of analytic rank 2 therein.

9.2.2. *Motivic weight 3, hypergeometric motives.* The symmetric cube examples do not have full Sato-Tate group (see [13] in this regard), and one might thus call them "deficient" in a similar sense that (say) imprimitive examples would be. In conjunction with Fernando Rodriguez Villegas and particularly David Roberts, we have computed [35, §18.5] central L-values for a number of hypergeometric motives, in particular all 487 degenerations at $t = 1$ of degree 6 weight 3 data. These give degree 4 L-functions of motivic weight 3, and we find 4 examples[28] of analytic rank 3 and 73 of analytic rank 2.

We have not made a similar exploration of degree 4 weight 3 hypergeometric data for other values of t, as it can be hard to compute the local information for wild primes. Another method to get L-functions of this degree and weight might be (following [13]) to take a tensor product of two degree 2 L-functions, one of motivic

[26]Note that the twist case lowers the probability, as the real-period reciprocal grows faster.

[27]The genus 2 record appears to be rank ≥ 26 by Elkies [11].

[28]There are actually 5 hypergeometric data that yield analytic rank 3 for the $t = 1$ degeneration, but only 4 distinct motives. Similarly, there are 80 that yield analytic rank 2.

weight 1 and the other of motivic weight 2. It does not seem that the technology for Siegel modular forms is sufficiently developed for truly "generic" examples to be computed to any great extent here.

9.2.3. *Motivic weight 5.* Again there are hypergeometric motives, namely taking the $t = 1$ degenerations of the 142 hypergeometric data of degree 6 and weight 5, and 35 of these data give a motive of analytic rank 2, for 34 distinct examples.

9.3. **Degree 6.** Again the (primitive) weight 1 examples should be Jacobians of genus 3 curves or elliptic curves over cubic fields, and we expect high ranks to occur. For weight 3, presumably some experiments with hypergeometric motives could be done, though again the question of wild prime data is difficult. With weight 5, the paper [24] considers Sym^5 for about[29] 8% of the elliptic curves of conductor up to 130000, and finds 569 examples of analytic rank 2 and 12 of analytic rank 3, the first of the latter being 816b, whose Sym^5 conductor is $2^{12}3^5 17^5 \approx 1.41 \cdot 10^{12}$.

9.4. **Higher degrees.** As always, the weight 1 examples will consist of curves, where we could expect largish ranks to be possible. For higher weight, I don't know of any data other than symmetric powers of elliptic curves in [24], though maybe David Roberts has computed a few sporadic cases (with a single wild prime) of $t = 1$ hypergeometric degenerations. Although the data of [24] are for quite limited objects, they are already interesting in that they give examples up to the 13th symmetric power (of 324c) with analytic rank 2, in contrast to [32, §6.5] which suggests extra vanishing should be rare already for 9th symmetric powers. Furthermore, the data from $t = 1$ degenerations of degree 6 hypergeometric data (thus degree 4 L-functions) already suggests a conjecture that for higher degrees there should be lots of examples of analytic rank 2 for all odd weights up through one more than the degree.[30] For instance, (Φ_2^8, Φ_4^4) at $t = 1$ has degree 6, weight 7, and conductor 2^{12}, with analytic rank 2. Another example is $(\Phi_2^{10}, \Phi_4^3\Phi_8)$ at $t = 1$ with degree 8, weight 9, and conductor 2^{22} again with analytic rank 2.

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