Affine Kac-Moody Lie Algebras in MAGMA

Dan Roozemond
Computational Algebra Group
University of Sydney

Computation in Coxeter groups and Kac-Moody algebras
Sydney, 16 November 2011

THE UNIVERSITY OF SYDNEY

MAGMA
COMPUTER • ALGEBRA

Serre relations

For A a generalized Cartan matrix, define $\mathfrak{g}'(A)$ to be the Lie algebra generated by 3n generators e_i, f_i, h_i and relations:

$$\begin{cases} [h_i, h_j] = 0, \quad [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \;\; (i \neq j) \\ [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\ (\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0. \;\; (i \neq j) \end{cases}$$

Three classes

(a) Finite-dimensional (if there is a positive integral vector v such that $Av > 0$);
(b) Affine (if there is a positive integral vector v such that $Av = 0$);
(c) Complicated (if there is a positive integral vector v such that $Av < 0$).

Three classes

(a) Finite-dimensional (if there is a positive integral vector v such that $Av > 0$);

(b) Affine (if there is a positive integral vector v such that $Av = 0$);

(c) Complicated (if there is a positive integral vector v such that $Av < 0$).

Non-twisted Affine Lie Algebras

For a generalized Cartan matrix A of type $X_l^{(1)}$, take $\mathfrak{g}_0$ to be the finite-dimensional Lie algebra of type X_l and take

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with multiplication on $\mathfrak{g}$ given by

$$\left[t^k \otimes x \oplus \lambda c \oplus \mu d, t^{k_1} \otimes y \oplus \lambda_1 c \oplus \mu_1 d\right] = \left(t^{k+k_1} \otimes [x, y] + \mu k_1 t^{k_1} \otimes y - \mu_1 k t^k \otimes x\right) \oplus k\delta_{k,-k_1}(x|y)c$$

Theorem [Kac90, Theorem 7.4]

$\mathfrak{g}$ is the affine Kac-Moody algebra associated to A.

Generators

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

- Arithmetic is very straightforward; except equality testing
- 2 normal forms:

$$(t^{-1} - 1 + t^2) \otimes (2E_1 + 3F_2) + \frac{1}{3}c$$

$$\begin{aligned} &t^{-1} \otimes (2E_1 + 3F_2) + \\ &-1 \otimes (2E_1 + 3F_2) + \\ &t^2 \otimes (2E_1 + 3F_2) + \\ &+\tfrac{1}{3}c \end{aligned}$$

$$\begin{aligned} &(2t^{-1} - 2 + 2t^2) \otimes E_1 + \\ &(3t^{-1} - 3 + 3t^2) \otimes F_2 + \\ &\tfrac{1}{3}c \end{aligned}$$

Demo

THE UNIVERSITY OF SYDNEY

Conclusion

- We can currently do a rather small number of things,
- but I’d be interested in what you would want it to do!