On Lie Algebras Generated by Few Extremal Elements

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The Definitions

A Lie algebra L is a vector space with a multiplication $[\cdot,\cdot]:L\times L\mapsto L$ that

- is bilinear [x+y,z] = [x,y] + [x,z]
- is anti-symmetric [x,y] = -[y,x]
- · satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y] = 0$$

Example: \mathfrak{sl}_3

The 3x3 matrices of trace 0, with [M,N] := MN - NM Basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

+6 off-diagonal ones, so $dim(\mathfrak{sl}_3)=8$

An element x of L is called extremal if, for all $y \in L$

$$[x,[x,y]] = \alpha_y x$$
 for some $\alpha_y \in k$.

Example: \mathfrak{sl}_3

$$[E_{12}, [E_{12}, h_1]] = 0$$

$$[E_{12}, [E_{12}, h_2]] = 0$$

$$[E_{12}, [E_{12}, E_{12}]] = 0$$

$$[E_{12}, [E_{12}, E_{23}]] = 0$$

$$[E_{12}, [E_{12}, E_{13}]] = 0$$

$$[E_{12}, [E_{12}, E_{21}]] = -2E_{12}$$

$$[E_{12}, [E_{12}, E_{32}]] = 0$$

$$[E_{12}, [E_{12}, E_{31}]] = 0$$

The Preliminaries

If x is extremal, then there is a bilinear form f_x such that

$$[x, [x, y]] = f_x(y)x$$

If L (over k) is generated by extremal elements, then

- · L has a basis of extremal elements
- There is a bilinear form $f:L\times L\to k$ with $f(x,y)=f_x(y)$

$$f(x,y) = f(y,x)$$

 $f(x,[y,z]) = f([x,y],z)$

Suppose L is generated by extremal elements x_1, \ldots, x_n with $f_{x_i} \equiv 0$ (aka sandwich elements). Then L is finite-dimensional ($d_n = \dim(L)$) and nilpotent.

Suppose L is generated by extremal elements x_1, \ldots, x_n . Then $\dim(L) \leq d_n$.

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The Setup

Graph Γ → Variety X — Lie algebras

Simple, finite, connected

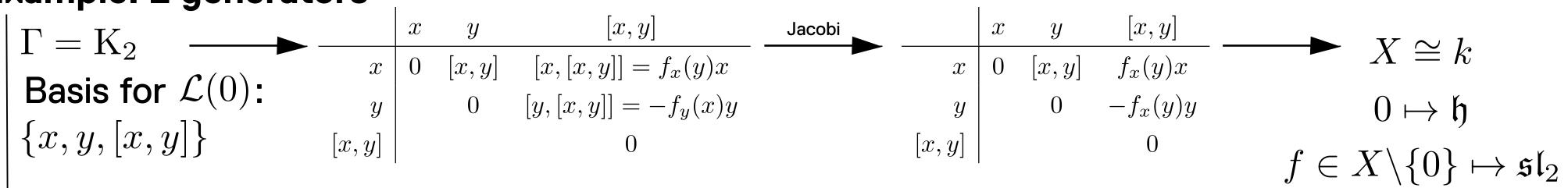
 $\Pi = V(\Gamma)$

$$\mathcal{F} = \langle x_1, \dots, x_n \rangle_{\text{Lie}} / \langle [x, y] \text{ for } x \not\sim y \rangle$$

For $f \in (\mathcal{F}^*)^{\Pi}$, notation $(f_x)_{x \in \Pi}$:

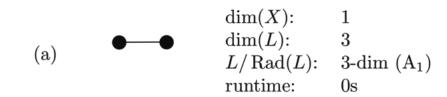
$$\mathcal{L}(f) := \mathcal{F}/\langle [x, [x, y]] - f_x(y)x \text{ for } x \in \Pi, y \in \mathcal{F} \rangle$$
$$X := \{ f \in (\mathcal{F}^*)^{\Pi} \mid \dim(\mathcal{L}(f)) = \dim(\mathcal{L}(0)) \}$$

Example: 2 generators



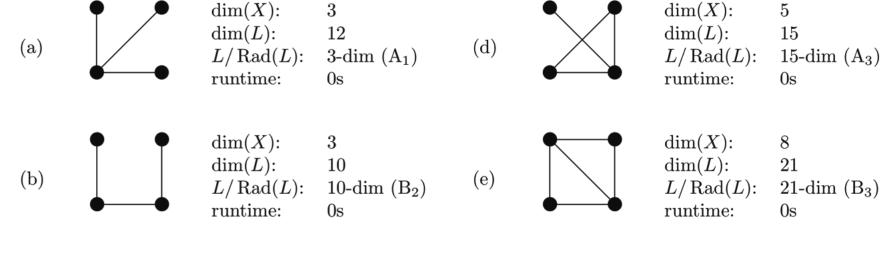
The Results

Using MAGMA for the computations, we determined $\mathcal{L}(0)$, X, and $\mathcal{L}(f)$ for all connected, simple, graphs Γ with at most 5 vertices.



(b)
$$\frac{\dim(X):}{\dim(L):} \quad 2 \\ \dim(L): \quad 6 \\ L/\operatorname{Rad}(L): \quad 3\text{-dim } (A_1) \\ \operatorname{runtime:} \quad 0s$$
 (c)
$$\frac{\dim(X):}{\dim(L):} \quad 4 \\ \dim(L): \quad 8 \\ L/\operatorname{Rad}(L): \quad 8\text{-dim } (A_2) \\ \operatorname{runtime:} \quad 0s$$

Table 1: Computational results (2 or 3 generators)



c)
$$\dim(X): \quad 5$$

$$\dim(L): \quad 15$$

$$L/\operatorname{Rad}(L): \quad 15\text{-dim } (A_3)$$

$$\operatorname{runtime:} \quad 0s$$

$$(f)$$

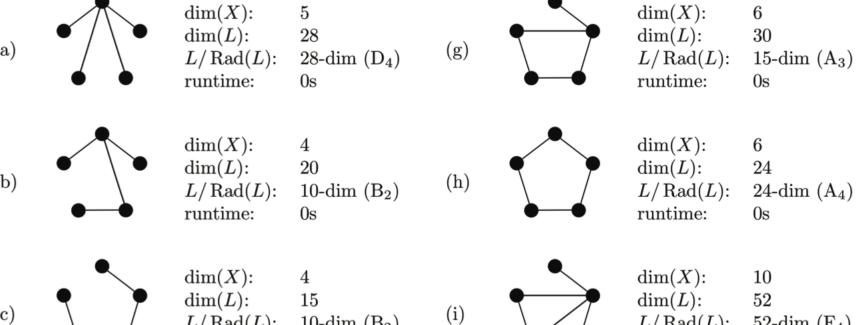
$$\lim(X): \quad 12$$

$$\dim(L): \quad 28$$

$$L/\operatorname{Rad}(L): \quad 28\text{-dim } (D_1)$$

$$\operatorname{runtime:} \quad 0s$$

Table 2: Computational results (4 generators)



$$\frac{\dim(L)\colon \quad 15}{L/\operatorname{Rad}(L)\colon \quad 10\text{-}\dim (B_2)} \qquad \text{(i)} \qquad \frac{\dim(L)\colon \quad 52}{L/\operatorname{Rad}(L)\colon \quad 52\text{-}\dim (F_1)}$$
runtime: 0s

$$\dim(X): \quad 7$$

$$\dim(L): \quad 36$$

$$L/\operatorname{Rad}(L): \quad 36\text{-dim } (B_4)$$

$$\operatorname{runtime:} \quad 0s$$

$$(j)$$

$$L/\operatorname{Rad}(L): \quad 45\text{-dim } (D)$$

$$\operatorname{runtime:} \quad 0s$$

$$\dim(X): \quad 9$$

$$\dim(L): \quad 45$$

$$L/\operatorname{Rad}(L): \quad 45-\dim (D_5)$$

$$\operatorname{runtime:} \quad 0s$$

$$\dim(X): \quad 12$$

$$\dim(L): \quad 134$$

$$L/\operatorname{Rad}(L): \quad 28-\dim (D_5)$$

$$\operatorname{runtime:} \quad 10s$$

$$\dim(X): \quad 13$$

$$\dim(X): \quad 21$$

$$\dim(X): \quad 21$$

$$\dim(X): \quad 13$$

(n)
$$\dim(X)$$
: 13 $\dim(L)$: 86 $\dim(L)$: 28-dim (D₄) (s) $\dim(L)$: 133 $\dim(L)$: 133-dim runtime: 14s $\dim(L)$: 14 $\dim(L)$: 78 $\dim(L)$

$$\dim(X): \quad 14$$

$$\dim(L): \quad 78$$

$$L/\operatorname{Rad}(L): \quad 78\text{-}\dim(E_6)$$

$$\operatorname{runtime:} \quad 2s$$

$$(u) \qquad \dim(X): \quad 0$$

$$\dim(L): \quad 537$$

$$L/\operatorname{Rad}(L): \quad \operatorname{trivial}$$

$$\operatorname{runtime:} \quad 38260s$$

(q) $\dim(X): \quad 14$ $\dim(L): \quad 78$ $L/\operatorname{Rad}(L): \quad 78\text{-dim } (E_6)$ $\operatorname{runtime}: \quad 2s$

Table 3: Computational results (5 generators)

- \Rightarrow In all cases, X is an affine space
- \Rightarrow K5 is *very* special: the only case where X is a point

The Algorithms

 $\mathcal{L}(f)$

- 1. Compute basis of $\mathcal{L}(0)$
 - Because we require maximum dimensionality of $\operatorname{our} \mathcal{L}(f)$'s, that is a basis $\operatorname{for} \mathcal{L}(f)$ (for all f in X) as well.
- 2. Compute "sufficient f-set"
 And let R be a corresponding
 "big" multivariate polynomial
 ring.
- 3. Compute multiplication table for $\mathcal{L}(f)$ over RUsing Premet identities, linear algebra, and tricks.
- 4. Compute "free f-set" Using the Jacobi identity; and if lucky thus prove $X\cong k^n$

MAGMA

MAGMA is a large computeralgebra system designed for computations in:

- Groups
- Semigroups and Monoids
- Rings and their Fields
- Global Arithmetic Fields
- Local Arithmetic Fields
- Linear Algebra and Module Theory
- Lattices and Quadratic Forms
- Associative Algebras
- Representation Theory
- Lie Theory
- Commutative Algebra
- Algebraic Geometry
- Arithmetic Geometry
- Modular Arithmetic Geometry
- Differential Galois Theory
- Geometry
- Combinatorial Theory
- Coding Theory
- Cryptography

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