On Lie Algebras Generated by Few Extremal Elements

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The Definitions

A *Lie algebra* L is a vector space with a multiplication $[\cdot,\cdot] : L \times L \mapsto L$ that

- is bilinear $[x+y,z] = [x,y]+[x,z]$
- is anti-symmetric $[x,y] = -[y,x]$
- satisfies the Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$$

Example: $\mathfrak{sl}_3$

The 3x3 matrices of trace 0, with

$$[M,N] := MN - NM$$

Basis:

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

+6 off-diagonal ones, so $\dim(\mathfrak{sl}_3) = 8$

An element x of L is called *extremal* if, for all $y \in L$

$$[x,[x,y]] = \alpha_y x$$

for some $\alpha_y \in k$.

Example: $\mathfrak{sl}_3$

$$\begin{aligned}
[E_{12},[E_{12},h_1]] &= 0\\
[E_{12},[E_{12},h_2]] &= 0\\
[E_{12},[E_{12},E_{12}]] &= 0\\
[E_{12},[E_{12},E_{23}]] &= 0\\
[E_{12},[E_{12},E_{13}]] &= 0\\
[E_{12},[E_{12},E_{21}]] &= -2E_{12}\\
[E_{12},[E_{12},E_{32}]] &= 0\\
[E_{12},[E_{12},E_{31}]] &= 0
\end{aligned}$$

The Preliminaries

If x is extremal, then there is a bilinear form f_x such that

$$[x,[x,y]] = f_x(y)x$$

If L (over k) is *generated* by extremal elements, then

- L has a *basis* of extremal elements
- There is a bilinear form $f : L \times L \to k$ with

$$f(x,y) = f_x(y)$$
$$f(x,y) = f(y,x)$$
$$f(x,[y,z]) = f([x,y],z)$$

Suppose L is generated by *extremal* elements $x_1,\ldots,x_n$ with $f_{x_i} \equiv 0$ (aka *sandwich* elements). Then L is finite-dimensional ($d_n = \dim(L)$) and nilpotent.

Suppose L is generated by extremal elements $x_1,\ldots,x_n$. Then $\dim(L) \leq d_n$.

The Setup

Graph Γ ⟶ **Variety** X ⟶ **Lie algebras**

Simple, finite, connected

$\Pi = \mathrm{V}(\Gamma)$

$$\mathcal{F} = \langle x_1,\ldots,x_n\rangle_{\mathrm{Lie}} / \langle [x,y] \text{ for } x \not\sim y\rangle$$

For $f \in (\mathcal{F}^*)^{\Pi}$, notation $(f_x)_{x\in\Pi}$:

$$\mathcal{L}(f) := \mathcal{F}/\langle [x,[x,y]] - f_x(y)x \text{ for } x \in \Pi, y \in \mathcal{F}\rangle$$

$$X := \{f \in (\mathcal{F}^*)^{\Pi} \mid \dim(\mathcal{L}(f)) = \dim(\mathcal{L}(0))\}$$

$\mathcal{L}(f)$

Example: 2 generators

$\Gamma = \mathrm{K}_2$ ⟶

Basis for $\mathcal{L}(0)$: $\{x,y,[x,y]\}$

	x	y	$[x,y]$
x	0	$[x,y]$	$[x,[x,y]] = f_x(y)x$
y		0	$[y,[x,y]] = -f_y(x)y$
$[x,y]$			0

Jacobi ⟶

	x	y	$[x,y]$
x	0	$[x,y]$	$f_x(y)x$
y		0	$-f_x(y)y$
$[x,y]$			0

⟶ $X \cong k$

$0 \mapsto \mathfrak{h}$

$f \in X\backslash\{0\} \mapsto \mathfrak{sl}_2$

The Results

Using MAGMA for the computations, we determined $\mathcal{L}(0)$, X, and $\mathcal{L}(f)$ for all connected, simple, graphs Γ with at most 5 vertices.

	dim(X)	dim(L)	$L/\mathrm{Rad}(L)$	runtime
(a)	1	3	3-dim (A_1)	0s
(b)	2	6	3-dim (A_1)	0s
(c)	4	8	8-dim (A_2)	0s

Table 1: Computational results (2 or 3 generators)

	dim(X)	dim(L)	$L/\mathrm{Rad}(L)$	runtime
(a)	3	12	3-dim (A_1)	0s
(b)	3	10	10-dim (B_2)	0s
(c)	5	15	15-dim (A_3)	0s
(d)	5	15	15-dim (A_3)	0s
(e)	8	21	21-dim (B_3)	0s
(f)	12	28	28-dim (D_4)	0s

Table 2: Computational results (4 generators)

	dim(X)	dim(L)	$L/\mathrm{Rad}(L)$	runtime
(a)	5	28	28-dim (D_4)	0s
(b)	4	20	10-dim (B_2)	0s
(c)	4	15	10-dim (B_2)	0s
(d)	7	36	36-dim (B_4)	0s
(e)	6	30	15-dim (A_3)	0s
(f)	6	24	24-dim (A_4)	0s
(m)	9	45	45-dim (D_5)	0s
(n)	13	86	28-dim (D_4)	3s
(o)	14	78	78-dim (E_6)	2s
(p)	14	78	78-dim (E_6)	2s
(q)	14	78	78-dim (E_6)	2s
(g)	6	30	15-dim (A_3)	0s
(h)	6	24	24-dim (A_4)	0s
(i)	10	52	52-dim (F_4)	0s
(j)	9	45	45-dim (D_5)	0s
(k)	9	45	45-dim (D_5)	0s
(l)	10	52	52-dim (F_4)	0s
(r)	12	134	28-dim (D_4)	10s
(s)	21	133	133-dim (E_7)	14s
(t)	21	249	78-dim (E_6)	2510s
(u)	0	537	trivial	38260s !

Table 3: Computational results (5 generators)

⇒ In *all* cases, X is an affine space

⇒ K5 is *very* special: the only case where X is a point

The Algorithms

1. Compute basis of $\mathcal{L}(0)$
   Because we require maximum dimensionality of our $\mathcal{L}(f)$'s, that is a basis for $\mathcal{L}(f)$ (for all f in X) as well.
2. Compute "sufficient f-set"
   And let R be a corresponding "big" multivariate polynomial ring.
3. Compute multiplication table for $\mathcal{L}(f)$ over R
   Using Premet identities, linear algebra, and tricks.
4. Compute "free f-set"
   Using the Jacobi identity; and if lucky thus prove $X \cong k^n$

MAGMA

MAGMA is a large computeralgebra system designed for computations in:

- Groups
- Semigroups and Monoids
- Rings and their Fields
- Global Arithmetic Fields
- Local Arithmetic Fields
- Linear Algebra and Module Theory
- Lattices and Quadratic Forms
- Associative Algebras
- Representation Theory
- Lie Theory
- Commutative Algebra
- Algebraic Geometry
- Arithmetic Geometry
- Modular Arithmetic Geometry
- Differential Galois Theory
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