On Lie algebras generated by few extremal elements

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DIAMANT Symposium, 26/27 May 2011

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MAGMA COMPUTER · ALGEBRA

Contents

Definition: Lie algebra

A Lie algebra L is a vector space with a multiplication $[\cdot,\cdot] : L \times L \to L$ that:

- is bilinear, $[x+y,z] = [x,z] + [y,z]$
- is anti-symmetric, $[x,y] = -[y,x]$
- satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

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Example: $\mathfrak{sl}_3$

The 3x3 matrices of trace 0, with

$$[M,N] := MN - NM$$

Basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and 6 off-diagonal ones, so $\dim(\mathfrak{sl}_3) = 8$.

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An element x of L is called *extremal* if, for all y in L

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for some $\alpha_y \in k$.

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Example: $\mathfrak{sl}_3$

$$\begin{aligned} [E_{12},[E_{12},h_1]] &= 0 \\ [E_{12},[E_{12},h_2]] &= 0 \\ [E_{12},[E_{12},E_{12}]] &= 0 \\ [E_{12},[E_{12},E_{23}]] &= 0 \\ [E_{12},[E_{12},E_{13}]] &= 0 \\ [E_{12},[E_{12},E_{21}]] &= -2E_{12} \\ [E_{12},[E_{12},E_{32}]] &= 0 \\ [E_{12},[E_{12},E_{31}]] &= 0 \end{aligned}$$

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More examples

Famous simple Lie algebras (restricting to the *classical* ones)

Classification (in char. 0) due to Killing, Cartan: A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2

- correspond to root systems,
- have Chevalley bases,
- have char. p equivalents.

Famous extremal elements (CSUW, 2001)

The long root elements are extremal! (and the extremal elements are long roots if char. is not 2 or 3)

Cohen, Steinbach, Ushirobira, Wales (2001)

If x is extremal, there is a linear form f_x such that $[x,[x,y]] = f_x(y)x$

If L (over k) is generated by extremal elements, then

- L has a *basis* of extremal elements,
- There is a bilinear form $f : L \times L \to k$ with:

$$f(x,y) = f_x(y)$$
$$f(x,y) = f(y,x)$$
$$f(x,[y,z]) = f([x,y],z)$$

Suppose L is generated by *sandwich* elements $x_1, \ldots, x_n$. Then L is finite-dimensional (say $\dim(L) = d_n$ and nilpotent.

Suppose L is generated by *extremal* elements $x_1, \ldots, x_n$. Then $\dim(L) \leq d_n$.

Some extremal facts

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$f_x \equiv 0$ or $[x,[x,L]] = 0$

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Our setup

Graph Γ

Simple, finite, connected

$$\Pi = \mathrm{V}(\Gamma)$$

$$\Pi = \{x_1, \ldots, x_n\}$$

Variety

X

Lie algebras

$\mathcal{L}(f)$

Our setup

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Lie algebras

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Our setup

Graph Γ

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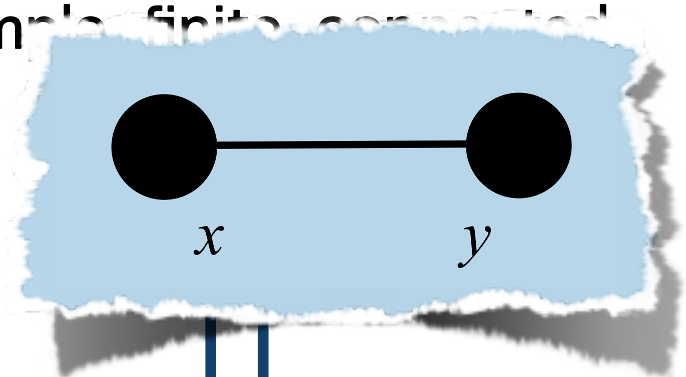

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Lie algebras $\mathcal{L}(f)$

Our setup

Graph Γ

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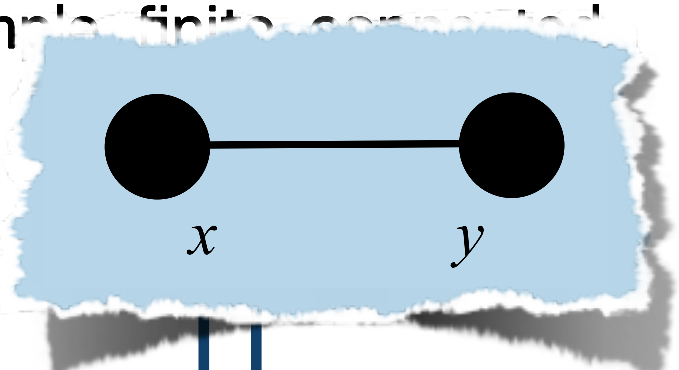

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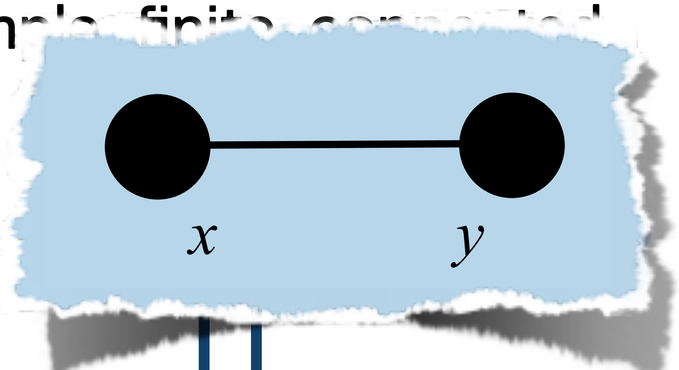

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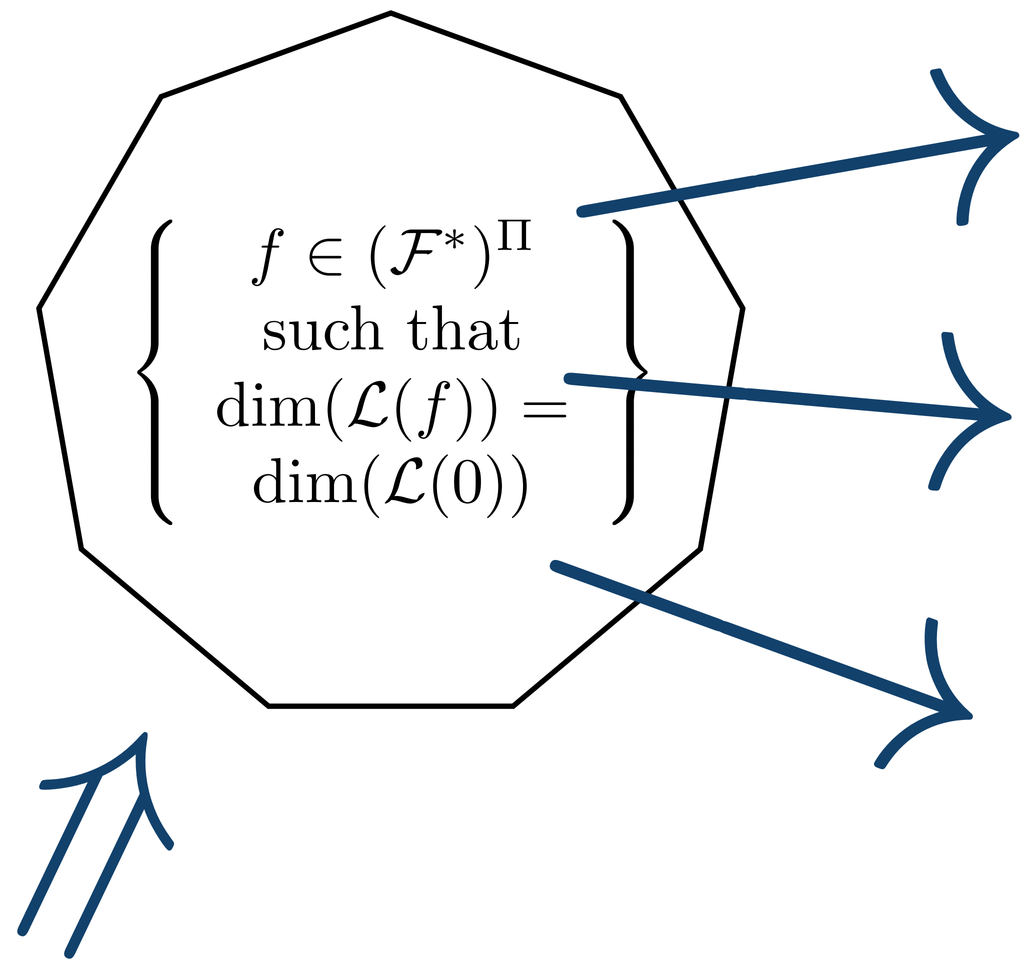


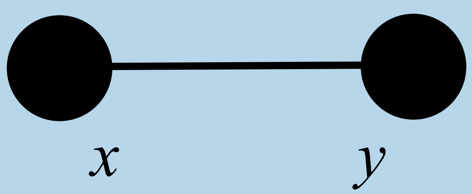
Our setup

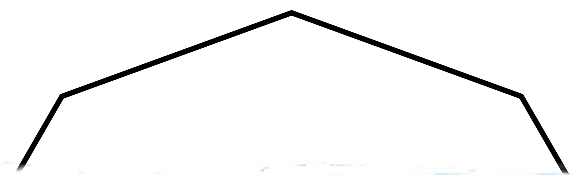
Graph Γ

Variety X

Lie algebras $\mathcal{L}(f)$

Simp... finite... conn...





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x	0	$[x,y]$	$[x,[x,y]] = f_x(y)x$
y		0	$[y,[x,y]] = -f_y(x)y$
$[x,y]$			0

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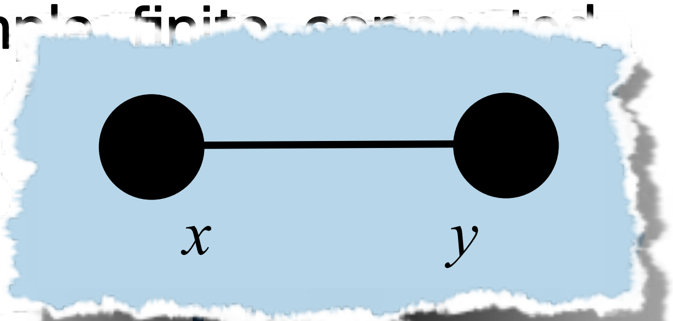

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$$[x,[y,[y,x]]] = f_y(x)[x,y]$$

$$\begin{aligned}[x,[y,[y,x]]] &= [y,[x,[y,x]]] + [[x,y],[y,x]] \\ &= -[y,[x,[x,y]]] + 0 \\ &= f_x(y)[x,y]\end{aligned}$$

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Graph Γ | Variety X | Lie algebras $\mathcal{L}(f)$

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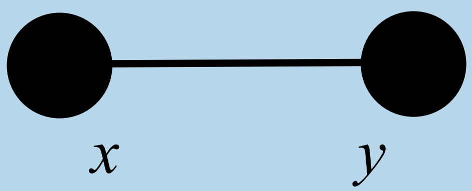

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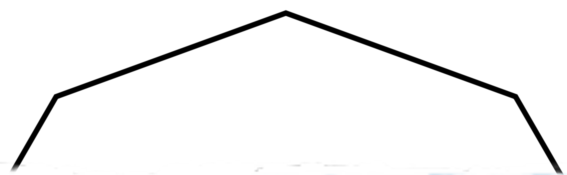

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&= f_x(y)[x,y]
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Our setup

Graph Γ

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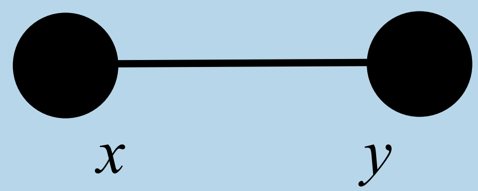

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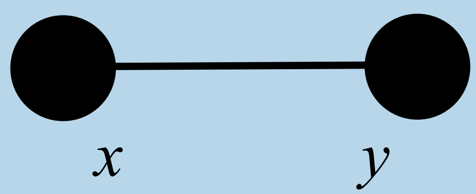

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Lie algebras $\mathcal{L}(f)$

$f \equiv 0 \mapsto \mathfrak{h}$

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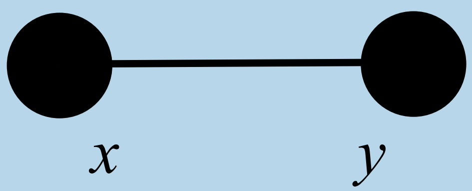

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Lie algebras
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$f \equiv 0 \mapsto \mathfrak{h}$

$f \in X \backslash \{0\} \mapsto \mathfrak{sl}_2$

	x	y	$[x, y]$
x	0	$[x, y]$	$f_x(y)x$
y		0	$-f_x(y)y$
$[x, y]$			0

Contents

Previous results

CSUW, 2001

$$
\begin{array}{llll}
\Gamma = \mathrm{K}_2: & \dim(\mathcal{L}(0)) = 3, & X \cong k, & \text{most } \mathcal{L}(f) \cong \mathrm{A}_1 \\
\Gamma = \mathrm{K}_3: & \dim(\mathcal{L}(0)) = 8, & X \cong k^4, & \text{most } \mathcal{L}(f) \cong \mathrm{A}_2 \\
\Gamma = \mathrm{K}_4: & \dim(\mathcal{L}(0)) = 28 & & \\
\Gamma = \mathrm{K}_5: & \dim(\mathcal{L}(0)) = 537 & &
\end{array}
$$

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Four series of graphs for each of the four classical series of Lie algebras

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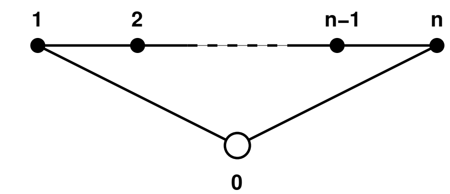

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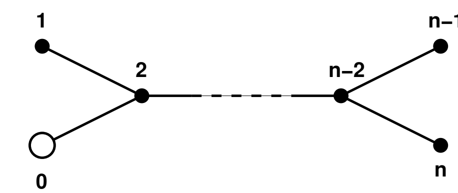

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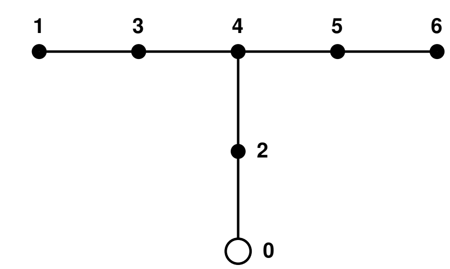
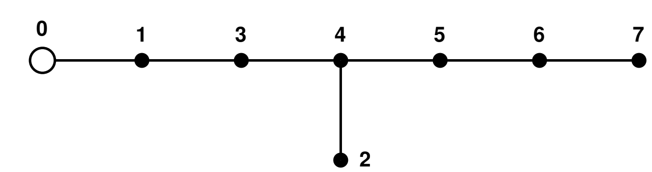
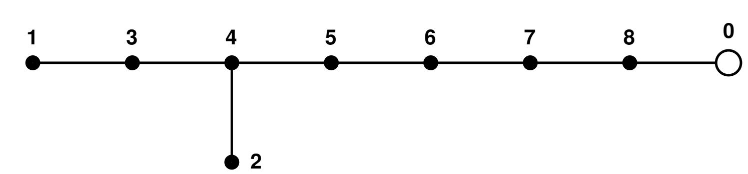


Draisma, in 't panhuis, 2008

- X is the set of k-rational points of an affine variety defined over k
- If Γ is a simply laced Dynkin diagram of affine type, then $X \cong k^{|\mathrm{E}(\Gamma)|}$ and, for f in an open dense subset of X, the Lie algebra $\mathcal{L}(f)$ is isomorphic to the Chevalley Lie algebra of type Γ^0





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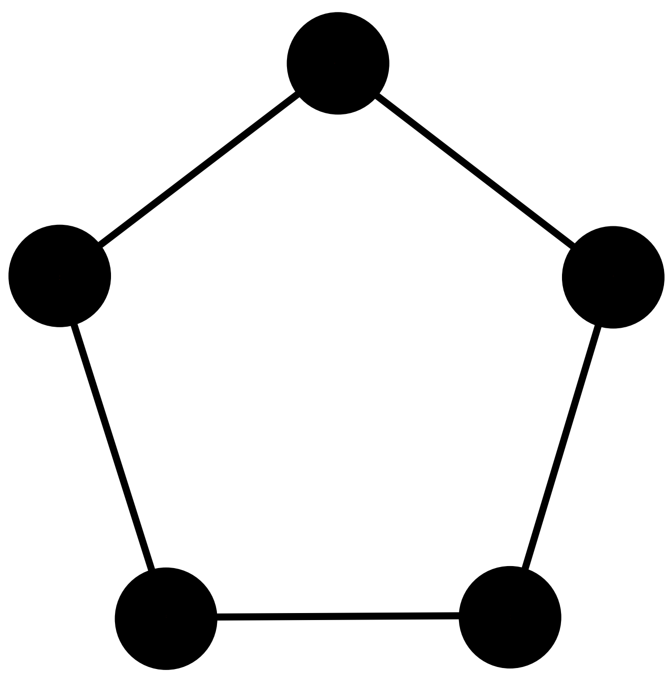

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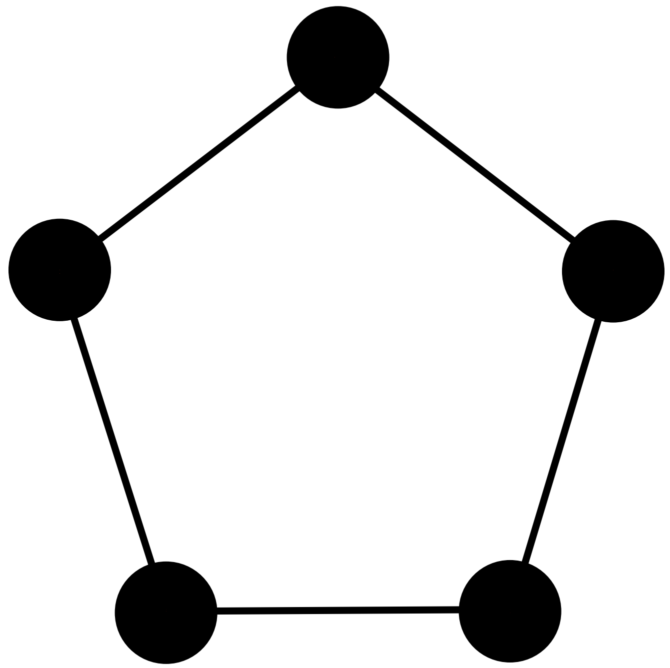

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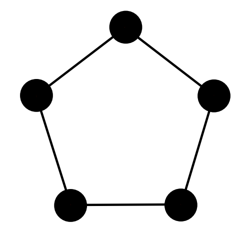

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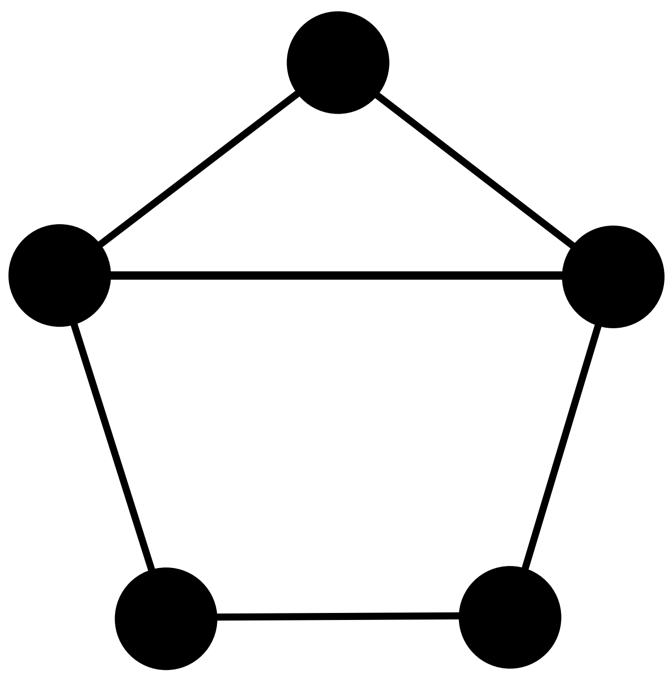

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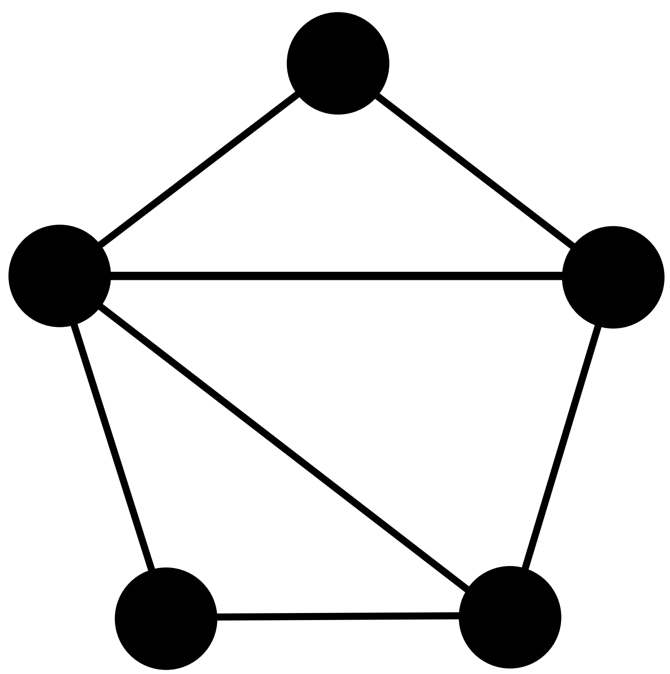

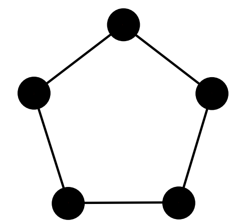

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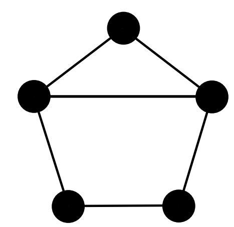

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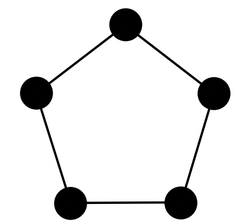

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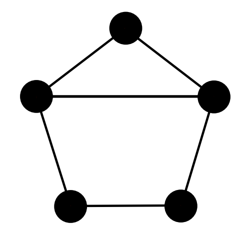

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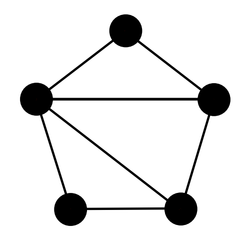

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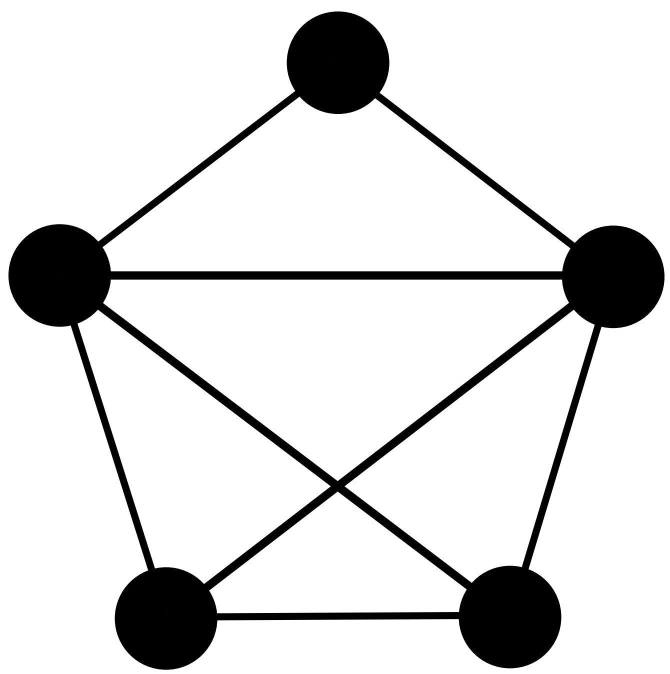

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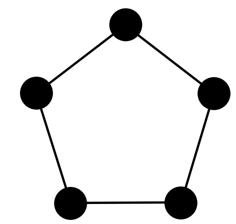

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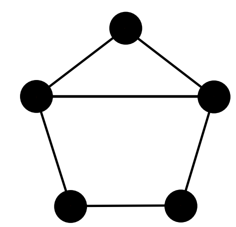

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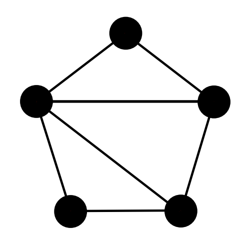

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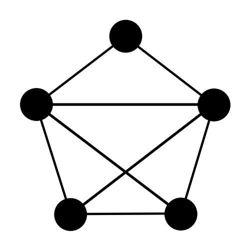

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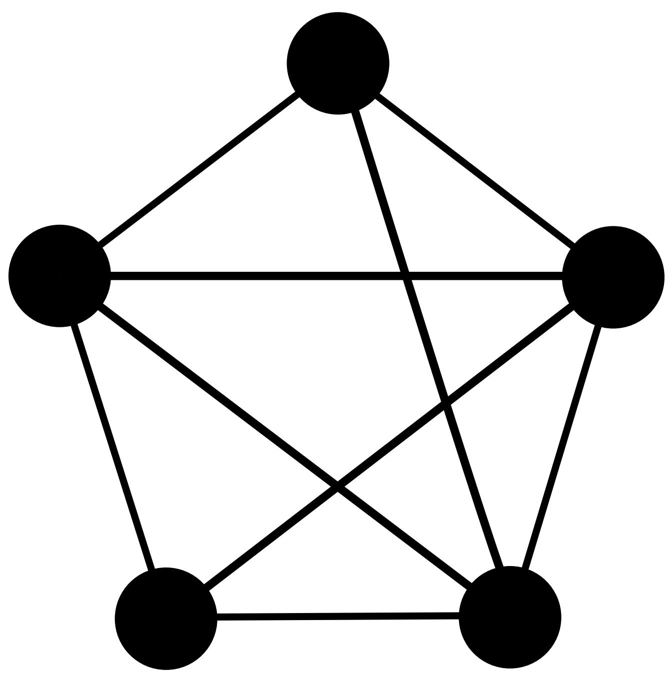

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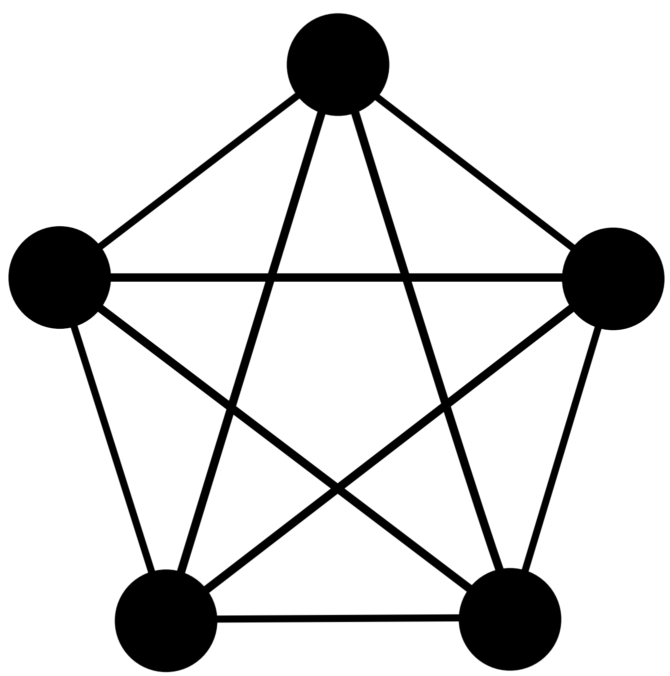

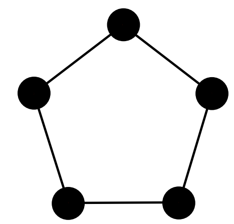

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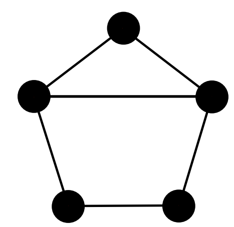

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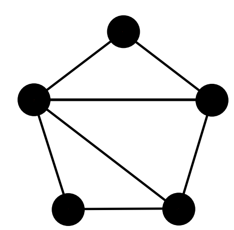

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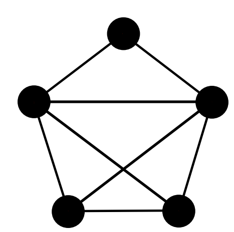

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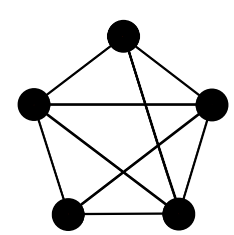

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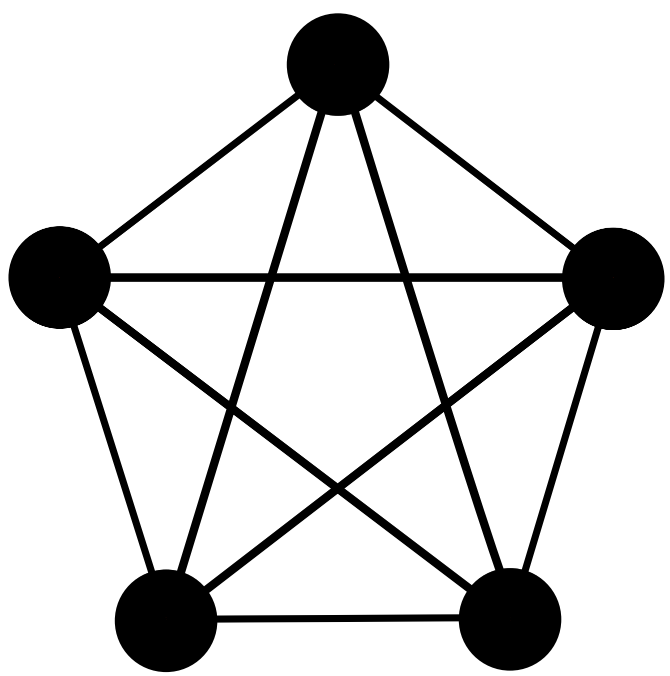

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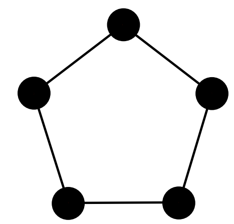

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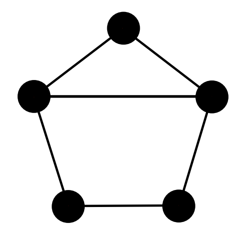

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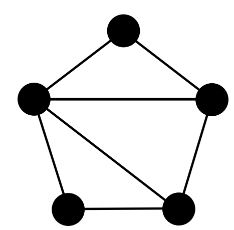

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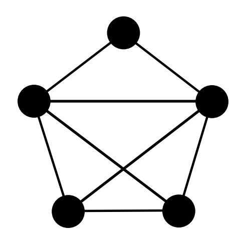

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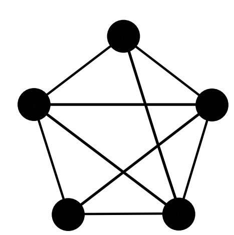

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- Semigroups and Monoids
- Rings and their Fields
- Global Arithmetic Fields
- Local Arithmetic Fields
- Linear Algebra and Module Theory
- Lattices and Quadratic Forms
- Associative Algebras
- Representation Theory
- Lie Theory
- Commutative Algebra
- Algebraic Geometry
- Arithmetic Geometry
- Modular Arithmetic Geometry
- Differential Galois Theory
- Geometry
- Combinatorial Theory
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- I'm here the week of June 6th and happy to come chat about MAGMA and new and old features!

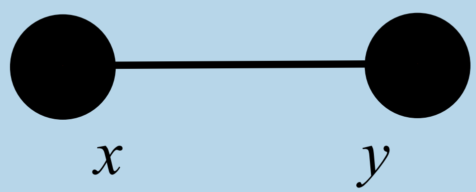
THE UNIVERSITY OF SYDNEY

Contents

Our setup

Graph Γ

Sin



$\Downarrow$

$$\mathcal{F} = \langle x_1, \ldots x_n \rangle_{\text{Lie}} / \langle [x,y] \text{ for } x \not\sim y \rangle$$

$\Downarrow$

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...not. $(f_x)_{x \in \Pi}$:

$\mathcal{L}(f) := \mathcal{F}/\langle[$ ... $\Pi, y \in$

$\mathcal{L}(0)$ " = " $\{x, y, [x,y]\}$

Variety X

$X \cong k$

$$\left\{ \begin{array}{c} f \in (\mathcal{F}^*)^{\Pi} \\ \text{such that} \\ \dim(\mathcal{L}(f)) = \\ \dim(\mathcal{L}(0)) \end{array} \right\}$$

Lie algebras $\mathcal{L}(f)$

$f \equiv 0 \mapsto \mathfrak{sl}_2$

$f \in X \backslash \{0\} \mapsto \mathfrak{h}$

	x	y	$[x,y]$
x	0	$[x,y]$	$[x,[x,y]] = f_x(y)x$
y		0	$[y,[x,y]] = -f_y(x)y$
$[x,y]$			0

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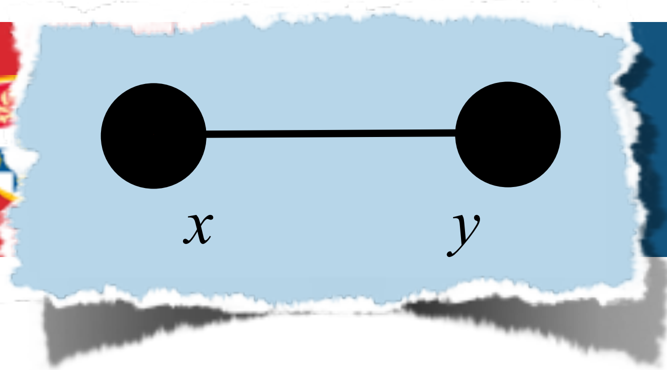

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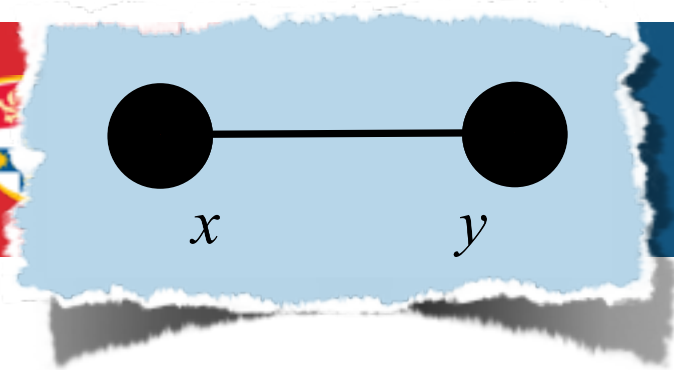

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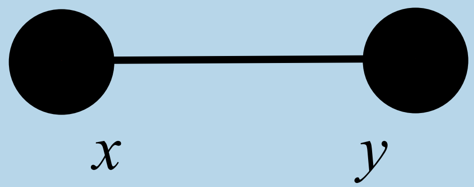

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$$R = k[\mathbf{fxx}, \mathbf{fxy}, \mathbf{fxxy}, \mathbf{fyx}, \mathbf{fyy}, \mathbf{fyxy}]$$

$$[x,[x,x]] = \mathbf{fxx} \cdot x$$

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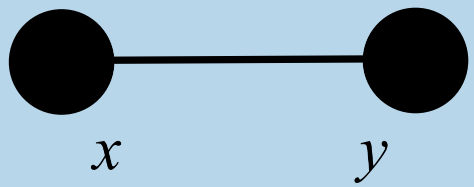


$\mathcal{L}(0)\text{“}=\text{”}\{x, y, [x,y]\}$

1. Compute basis of $\mathcal{L}(0)$

   Because we require maximum dimensionality of our $\mathcal{L}(f)$‘s, that is a basis for $\mathcal{L}(f)$ (for all f in X) as well.

$$R = k[\mathbf{fxx}, \mathbf{fxy}, \mathbf{fxxy}, \mathbf{fyx}, \mathbf{fyy}, \mathbf{fyxy}]$$

$$[x,[x,x]] = \mathbf{fxx} \cdot x$$

$$[x,[x,y]] = \mathbf{fxy} \cdot x$$

$$[x,[x,[x,y]]] = \mathbf{fxxy} \cdot x$$

$$\ldots$$

2. Compute “sufficient f-se

   And let R be a corresp

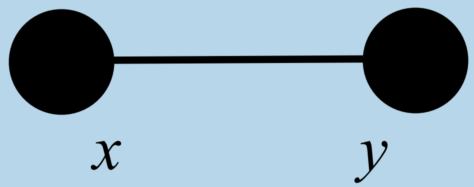
	x	y	$[x,y]$
x	0	$[x,y]$	$\mathbf{fxy} \cdot x$
y		0	$-\mathbf{fyx} \cdot y$
$[x,y]$			0

3. Compute multiplication table for $\mathcal{L}(f)$ over R

   Using Premet identities, linear algebra, and tricks.

4. Compute “free f-set”

   Using the Jacobi identity, and if lucky, thus prove that $X \cong k^n$



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4. Compute “free f-set”

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$$[y,[x,[x,y]]] + [x,[[x,y],y]] + [[x,y],[y,x]] + 0 = 0$$

$$-\mathbf{fxy}\cdot[x,y] + \mathbf{fyx}\cdot[x,y] = 0$$

$$\mathbf{fxy} = \mathbf{fyx}$$

proving $X \cong k^1$

Proving?

$\Gamma = \mathrm{K}_5$: $f(x_1, x_2) = 0$ **because**

f(1, [2]) -> f(2,[1]) {f(x,y) = f(y,x)}f(2, [1]) -> 0{[1,[1,[54352432514321]]] = (3/256*f21*f51*f32*f52*f43^2*f154 - 1/256*f21*f31*f52^2*f43^2*f154 - 1/256*f21*f31*f42*f52*f43*f53*f154 + 9/256*f21*f41*f32*f42*f53^2*f154 + 1/128*f21*f31*f42^2*f53^2*f154- 1/256*f21*f51*f32^2*f43*f54*f154 +
1/128*f21*f31*f32*f52*f43*f54*f154 + 3/256*f21*f41*f32^2*f53*f54*f154 + 9/256*f21*f31*f32*f42*f53*f54*f154 - 3/256*f21*f31*f32^2*f54^2*f154+ 1/32*f21*f41*f51*f32*f43*f53*f254 - 1/128*f21*f31*f41*f52*f43*f53*f254 - 1/128*f21*f31*f41*f42*f53^2*f254 - 1/64*f21*f31*f51*f32*f43*f54*f254 -
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Friday, 27 May 2011

...ving?

Musings

All cases considered: X is an affine space!

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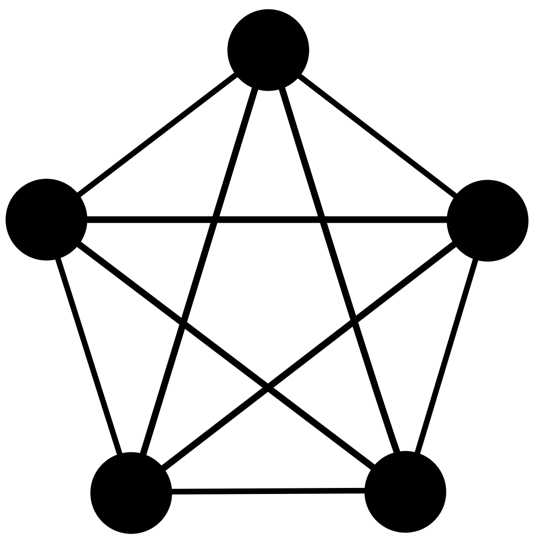

The first case where X is just a point

? Does this imply that K_n has trivial X if $n \geq 5$

? Is there a connection with graph planarity

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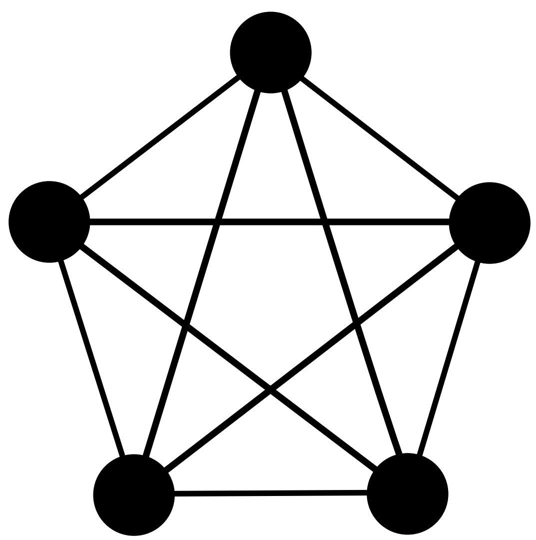

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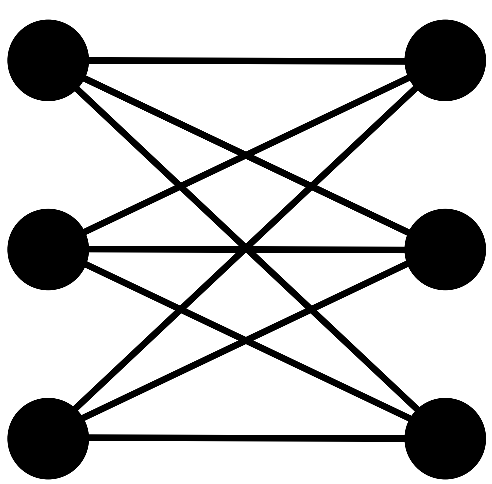

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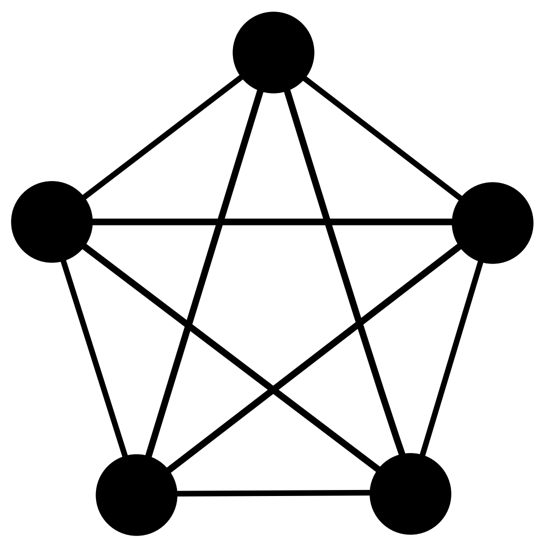

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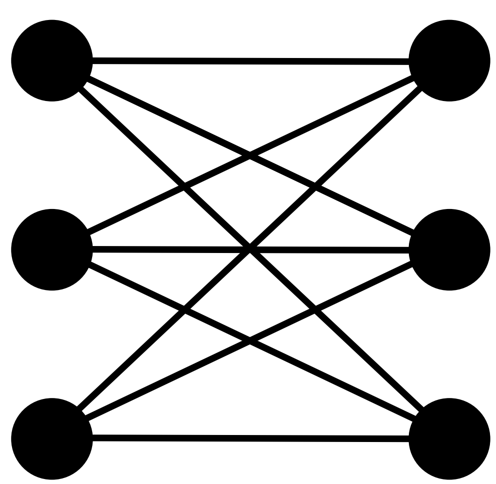

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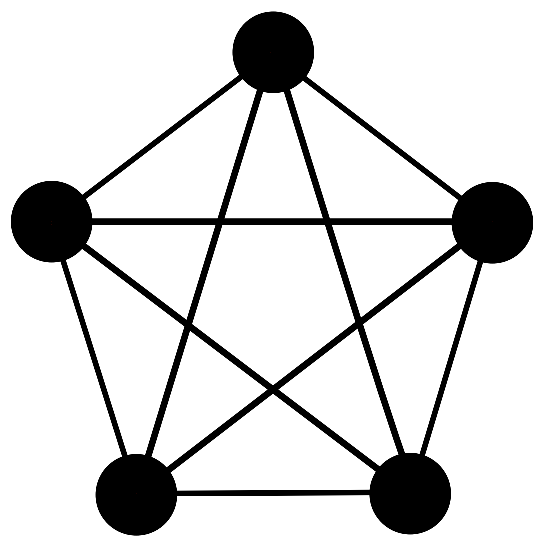

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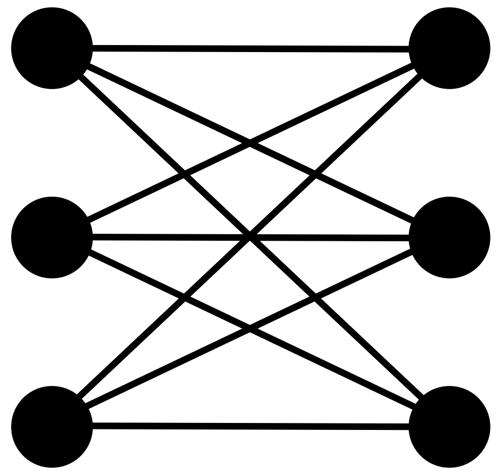

$\dim(\mathcal{L}(0)) = 969$
$\dim(X) = 0$

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