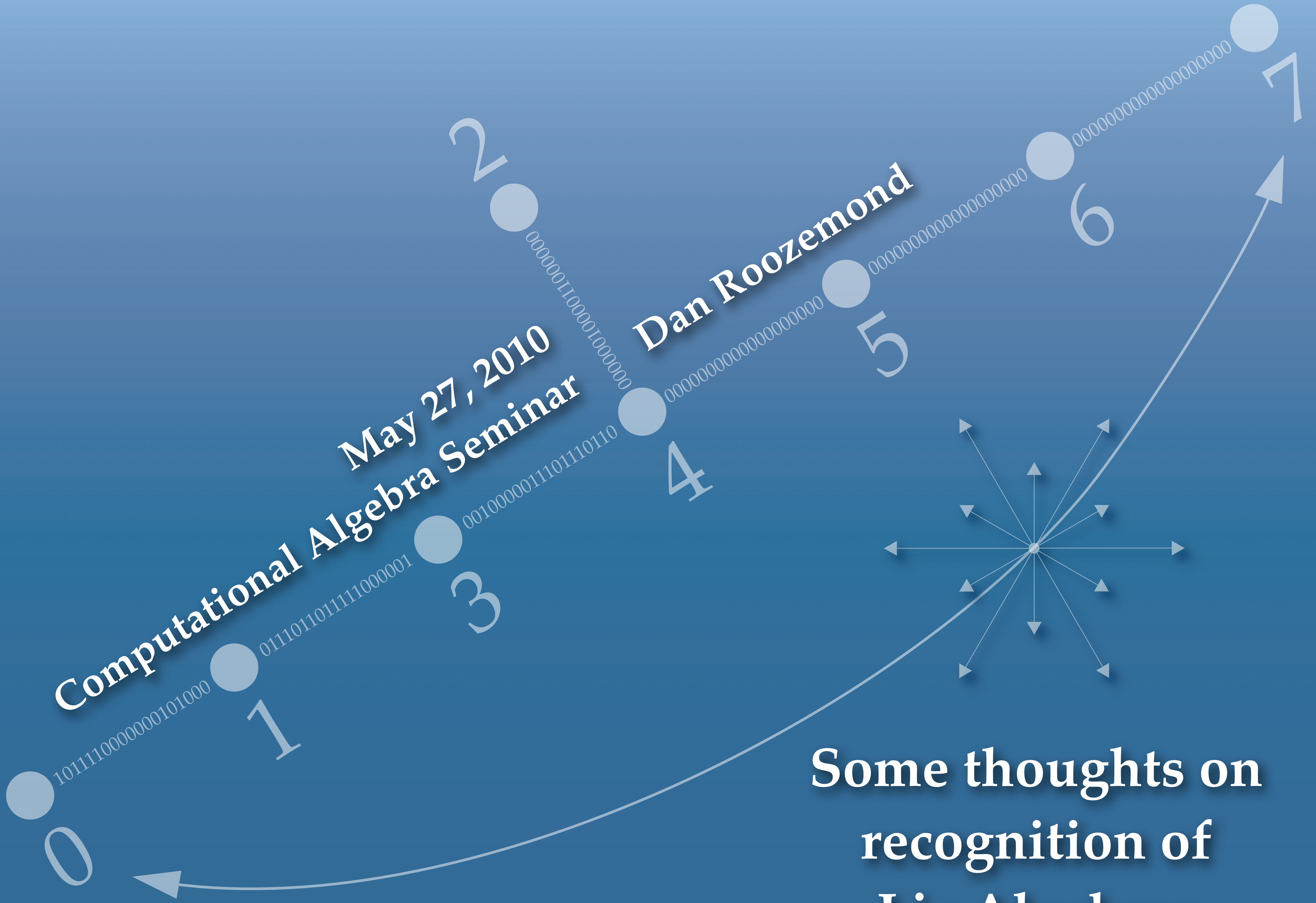
May 27, 2010

Computational Algebra Seminar

Dan Roozemond



Some thoughts on recognition of Lie Algebras

Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- What does 537 mean?

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What is a Lie Algebra?

Definition (*Lie algebra L*)

- Vector space $\mathbb{F}^n$
- With a multiplication $[\cdot,\cdot] : L \times L \to L$ that is
  - Bilinear,
  - Anti-symmetric, and
  - Satisfies the *Jacobi* identity

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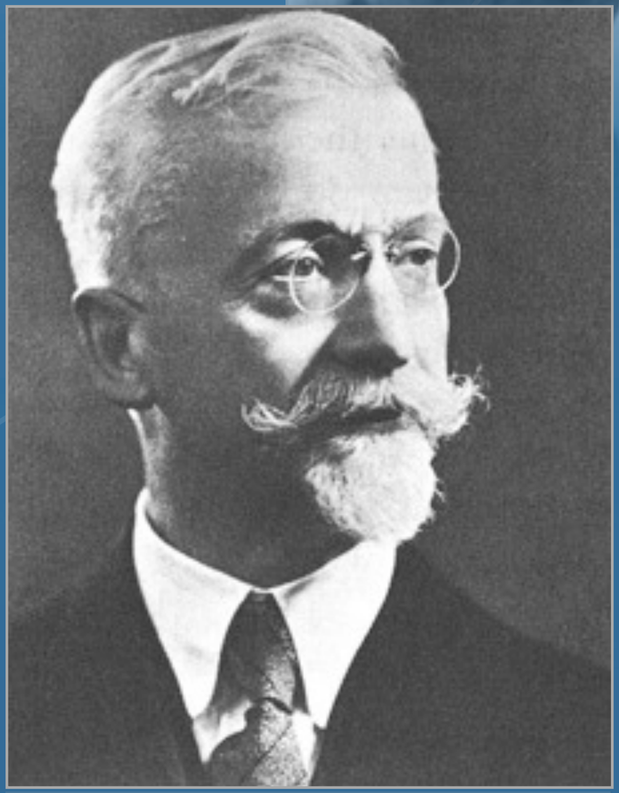
Simple Lie algebras

Classification (Killing, Cartan)

For $\mathbb{F}$ algebraically closed and $\mathrm{char}(\mathbb{F}) = 0$ the only simple Lie algebras are:

$$\begin{array}{lll} \mathrm{A}_n & (n \geq 1) & \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8 \\ \mathrm{B}_n & (n \geq 2) & \mathrm{F}_4 \\ \mathrm{C}_n & (n \geq 3) & \mathrm{G}_2 \\ \mathrm{D}_n & (n \geq 4) & \end{array}$$



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B_n	$(n \geq 2)$	F_4
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Classification (Premet, Strade)

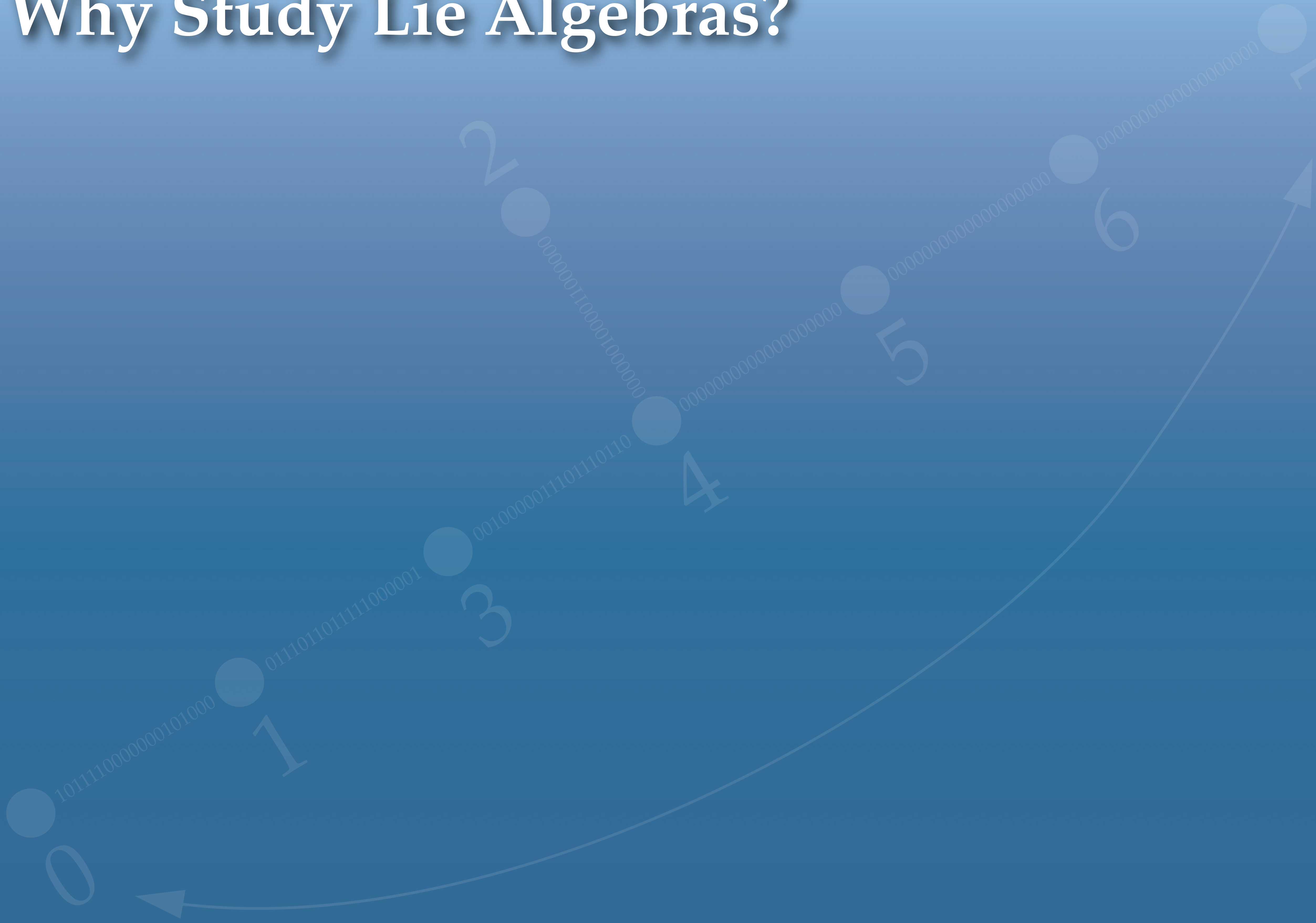
Over (alg. closed) finite fields:

- ● These *classical* Lie algebras,
- ✦ *Cartan type* Lie algebras,
- ✦ *Melikyan* Lie algebras.

... provided $\text{char}(\mathbb{F}) \geq 5$.

Other cases are WIP (Vaughan-Lee, Eick)

Why Study Lie Algebras?



Why Study Lie Algebras?

- Study *groups* by their Lie algebras:
  - Simple algebraic group G ↔ unique Lie algebra L
  - Many properties carry over to L
  - Easier to calculate in L
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- Opportunities for:
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  - ...
- Because there are problems to be solved!
  - ... and there was a thesis to be written...

(Almost trivial) Example (1)

- *Matrix* Lie algebra: elements are 2x2 matrices of trace 0, called $\mathfrak{sl}_2$
- Basis: $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$
- Multiplication: $[x, y] = xy - yx$

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Can turn any matrix algebra into a Lie algebra

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(Almost trivial) Example (2)

- As a *structure constant* algebra:
- Basis: e, f, h
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$[\cdot,\cdot]$	e	f	h
e	0	h	$-2e$
f	$-h$	0	$2f$
h	$2e$	$-2f$	0

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$$[E,H] = EH - HE = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2E$$

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$\mathfrak{sl}_2$!

aka A_1

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$$[e,[f,h]] + [f,[h,e]] + [h,[e,f]]$$

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$$\begin{aligned}&[e,[f,h]]+[f,[h,e]]+[h,[e,f]]\\&=[e,2f]+[f,-2e]+[h,h]\\&=2h+2h+0\neq 0\end{aligned}$$

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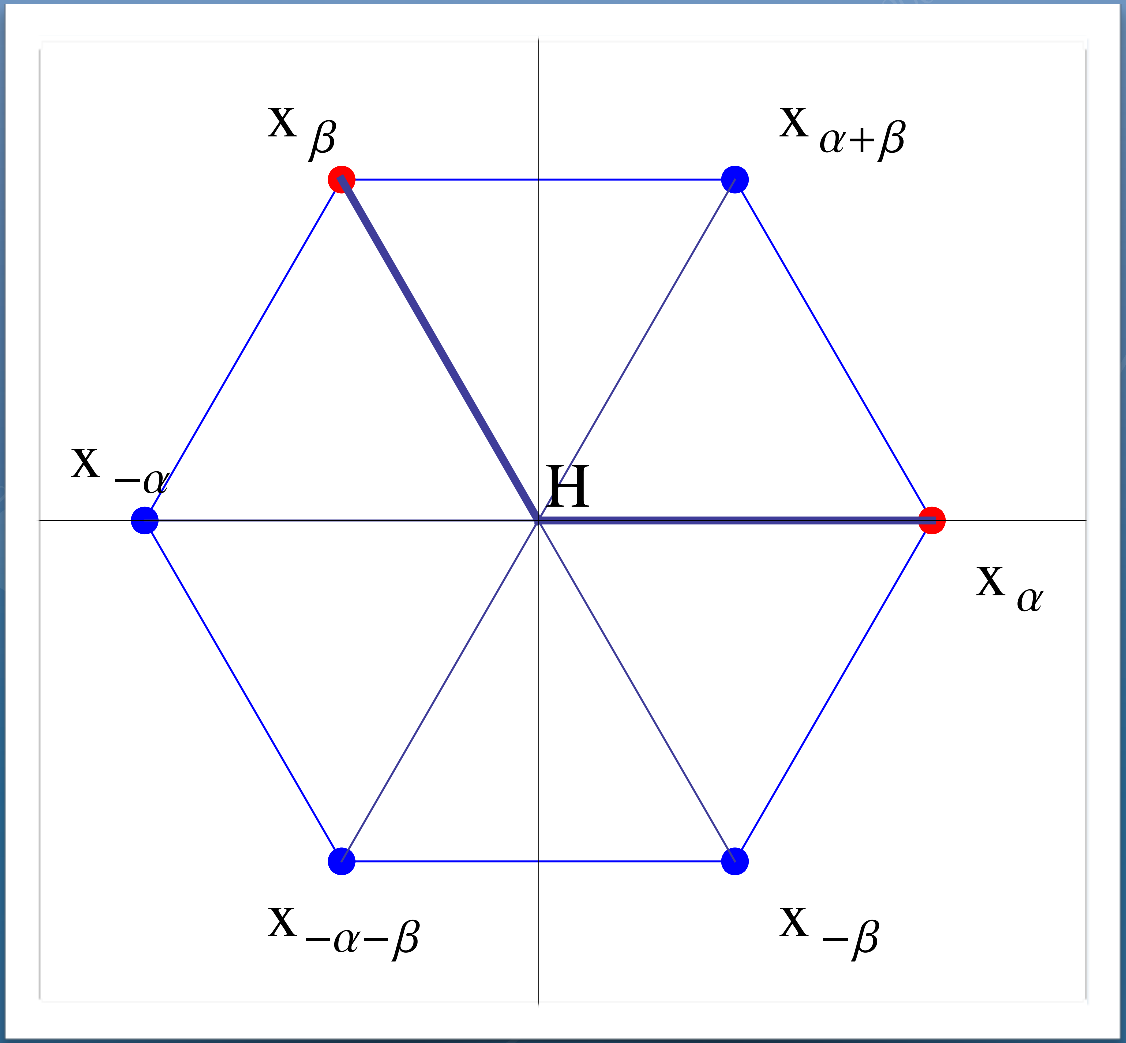
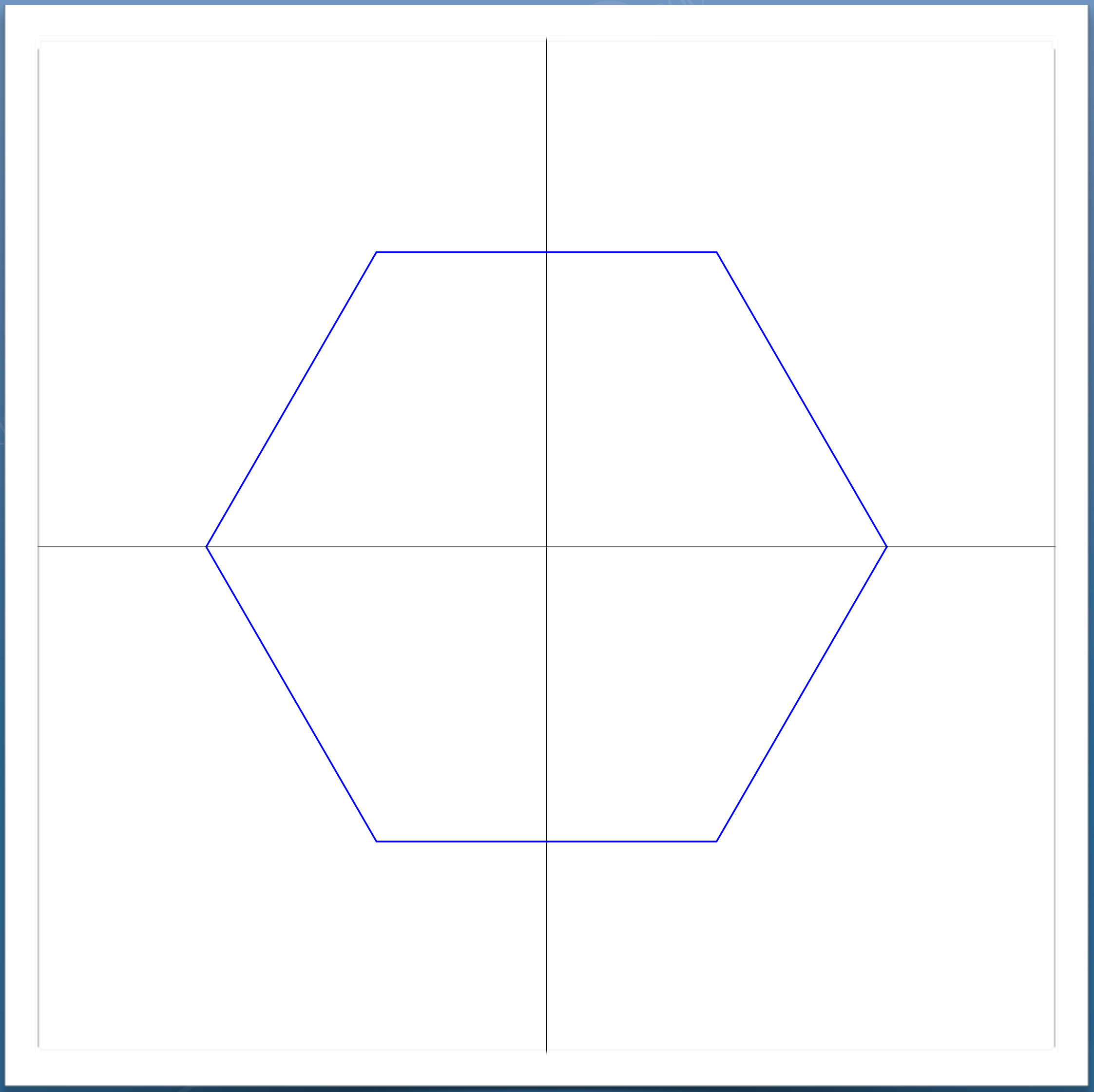
Chevalley Bases



Many Lie algebras have a *Chevalley basis!*

Root Systems

- A hexagon,



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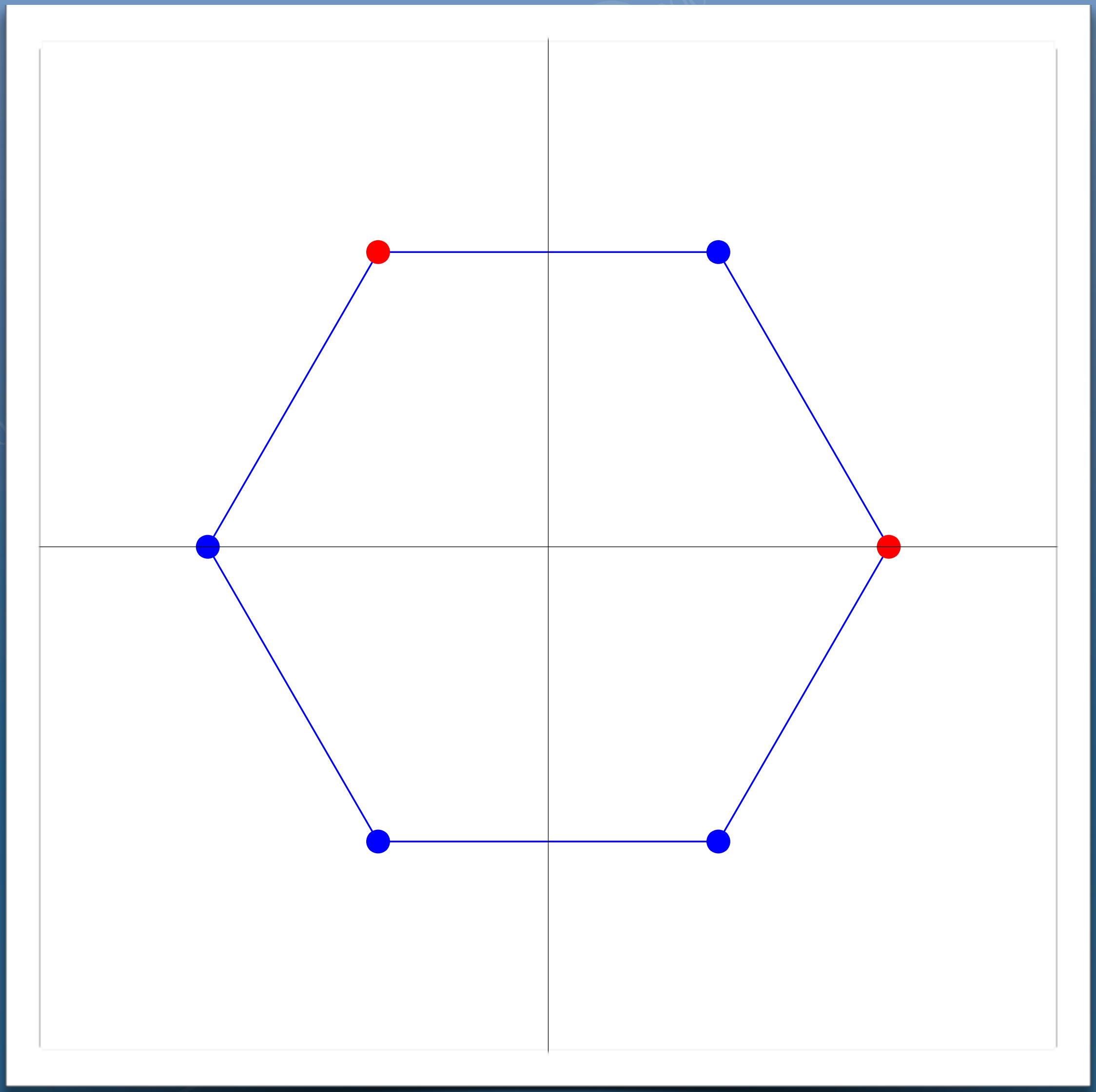

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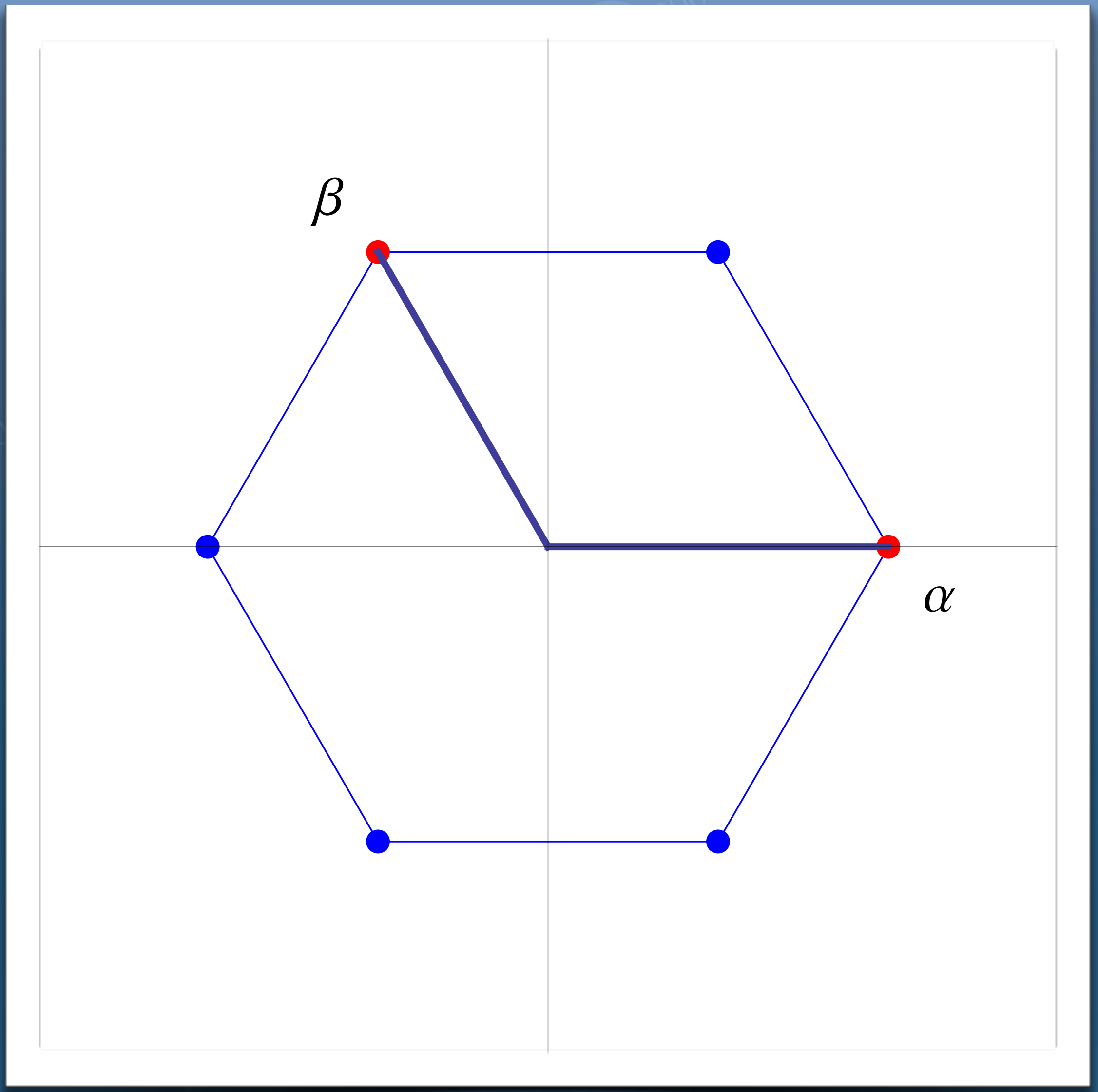
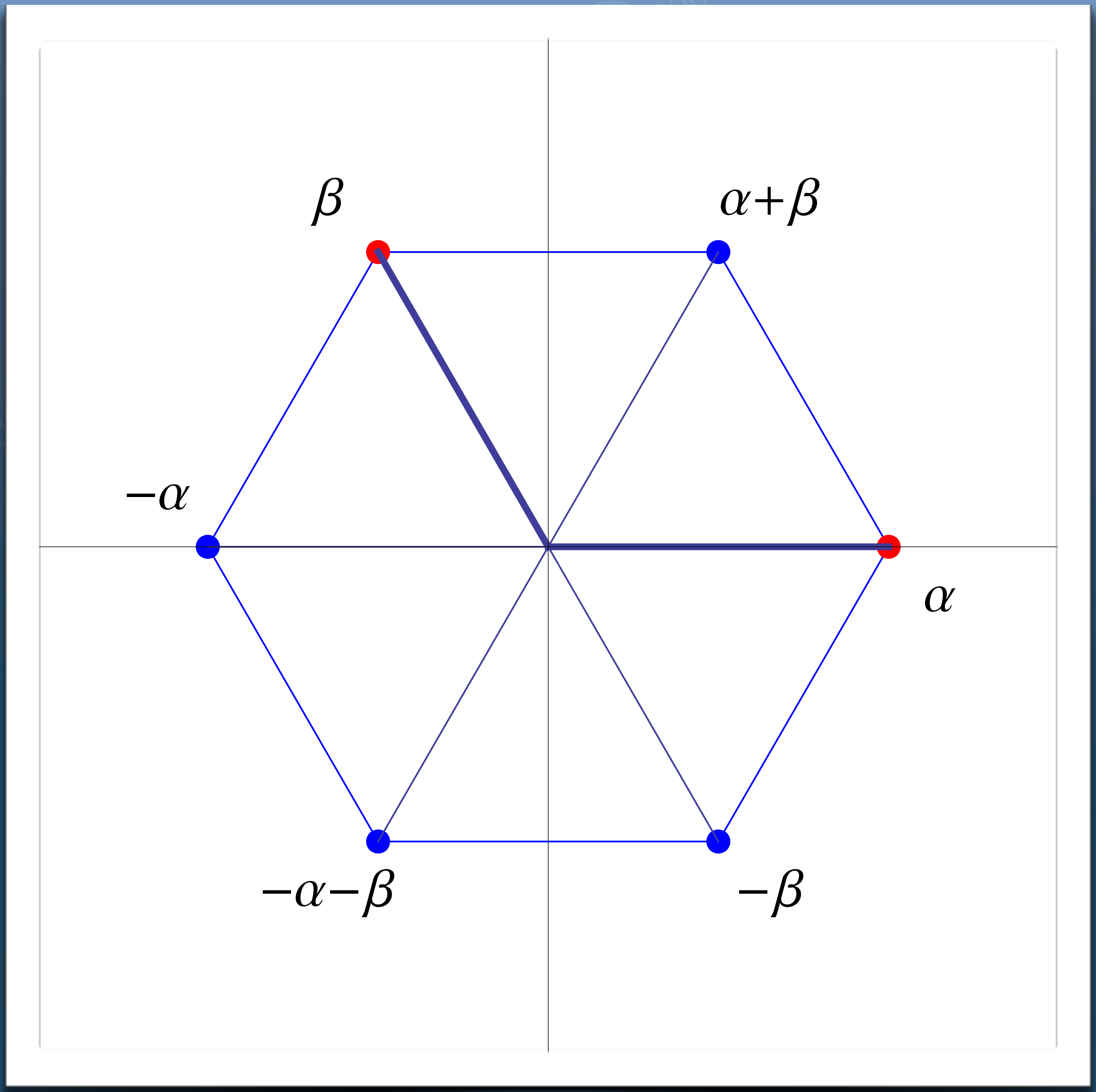


Root Systems

- A hexagon,
- A root system of type A_2



Root Data

Definition (*Root datum R*)

- $R = (X, \Phi, Y, \Phi^\vee), \quad \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$
- X and Y: Dual free $\mathbb{Z}$-modules,
- Put in duality by $\langle \cdot, \cdot \rangle$,
- $\Phi \subseteq X$: roots,
- $\Phi^\vee \subseteq Y$: coroots.

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One Root system $\longrightarrow$ $\left\{ \begin{array}{c} \textit{Several}\text{ Root data:} \\ \textit{“adjoint”} \\ \vdots \\ \textit{“simply connected”} \end{array} \right.$

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Irreducible root data: $A^{\cdot}_{n}, B^{\cdot}_{n}, C^{\cdot}_{n}, D^{\cdot}_{n}, E^{\cdot}_{6}, E^{\cdot}_{7}, E_{8}, F_{4}, G_{2}$.

Root Data

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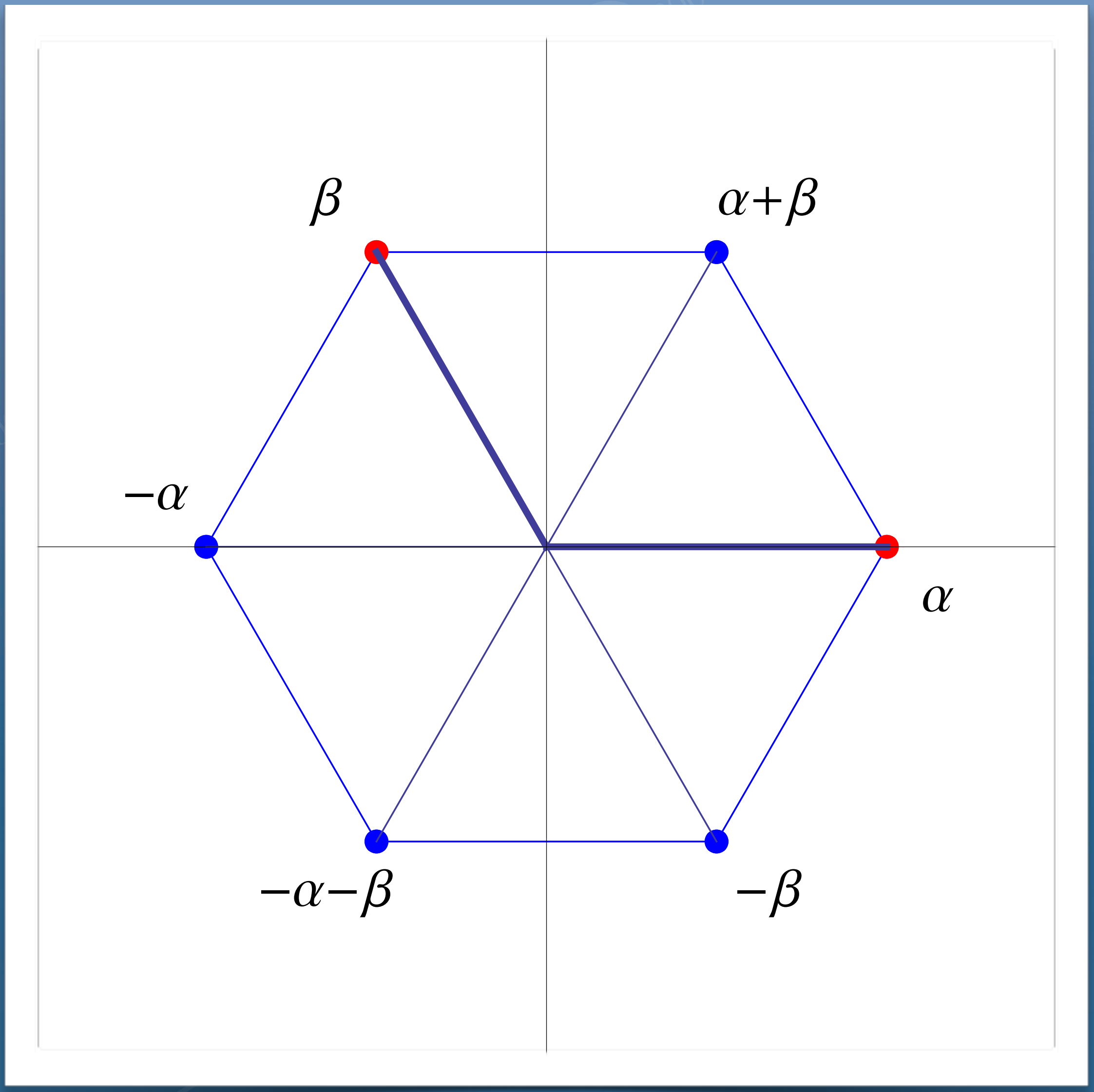
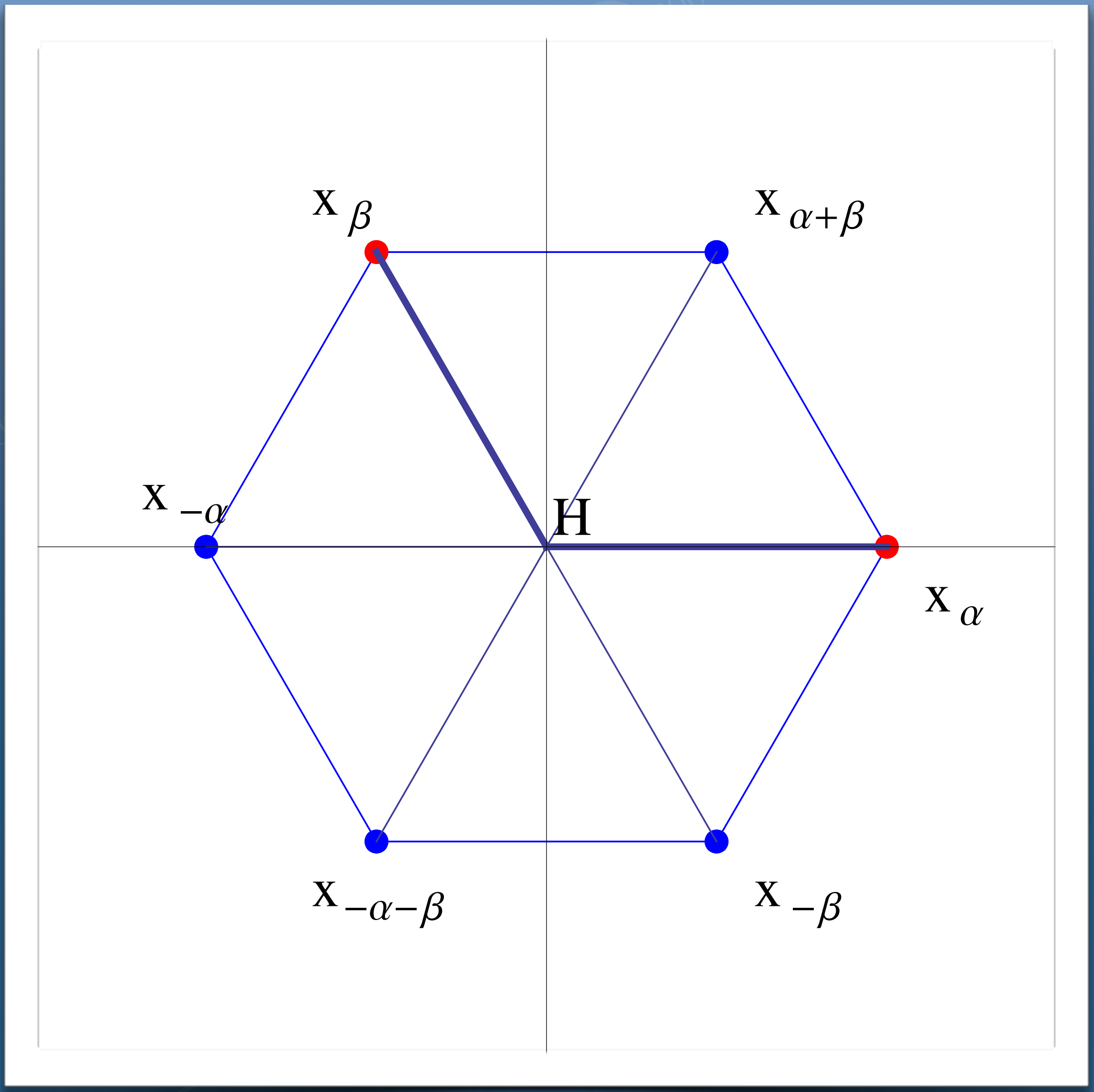


Root Data

- A hexagon,
- A root system of type A_2
- A Lie algebra of type A_2



Chevalley Basis

Definition (*Chevalley Lie algebra*)

- Basis: $L = \bigoplus_{i=1,\ldots,n} \mathbb{F}h_i \oplus \bigoplus_{\alpha\in\Phi} \mathbb{F}x_\alpha$
- Multiplication (for $i, j \in \{1, \ldots, n\}, \alpha, \beta \in \Phi$ and linearly extended) :

$$\begin{aligned} [h_i, h_j] &= 0, \\ [x_\alpha, h_i] &= \langle\alpha, f_i\rangle x_\alpha, \\ [x_{-\alpha}, x_\alpha] &= \textstyle\sum_{i=1}^{n} \langle e_i, \alpha^\vee\rangle h_i, \\ [x_\alpha, x_\beta] &= \begin{cases} N_{\alpha\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta\in\Phi, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and satisfying the Jacobi identity.

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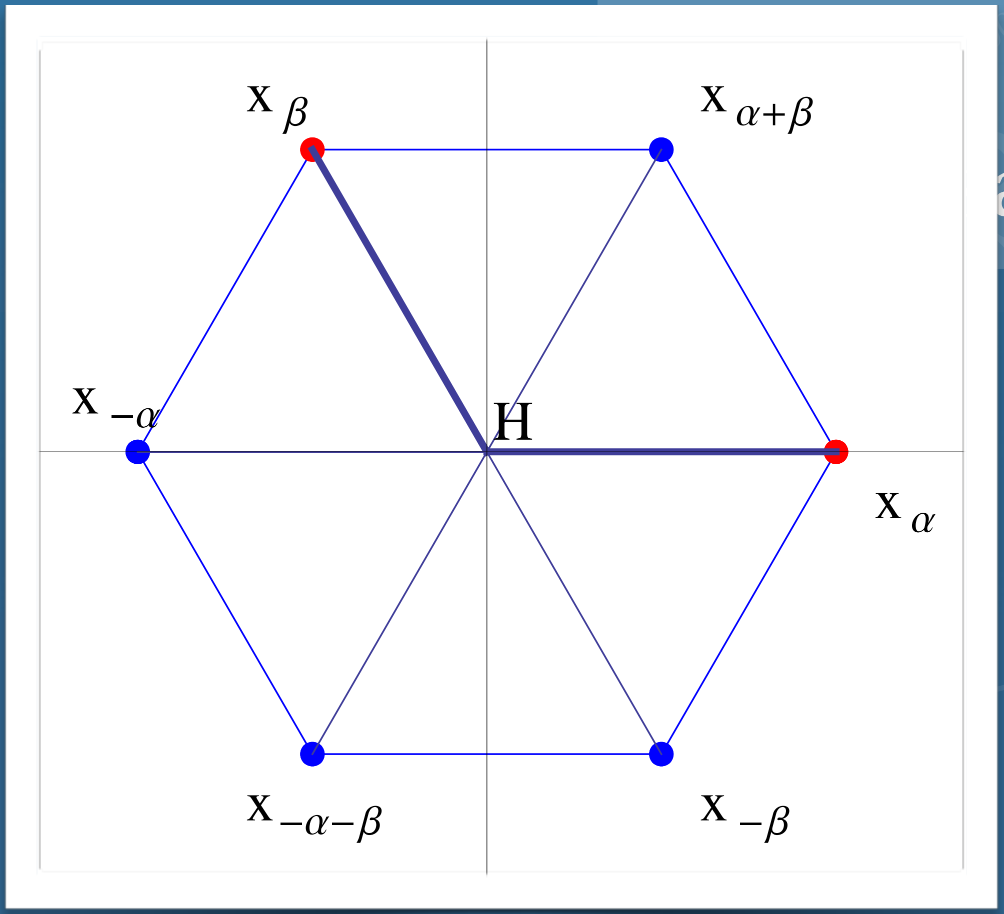

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Such a basis is called a *Chevalley basis*.

Why Chevalley bases?

- Because transformation between two Chevalley bases is an automorphism of L,
- So we can test isomorphism between two Lie algebras (and find isomorphisms!) by computing Chevalley bases.

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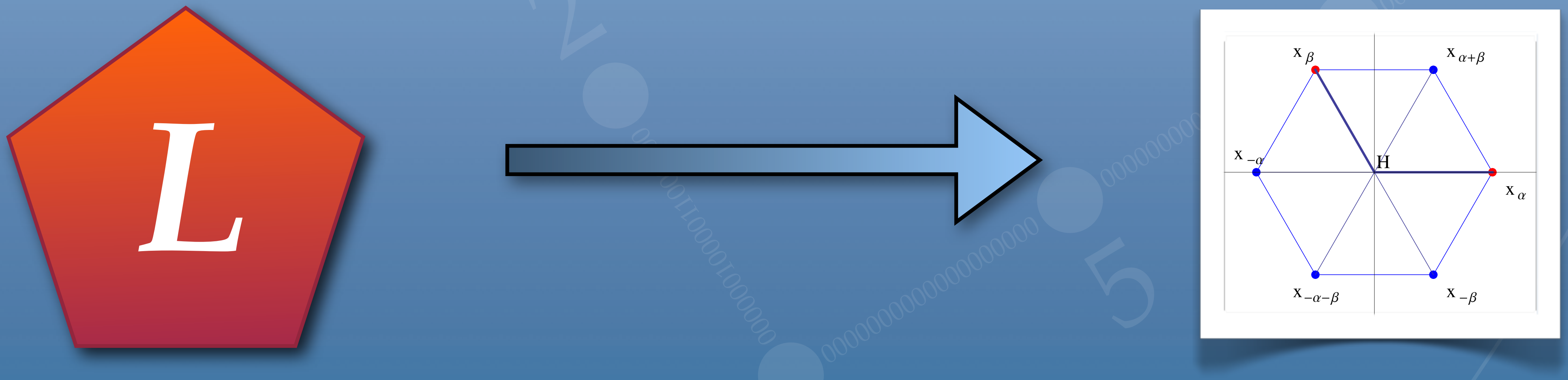

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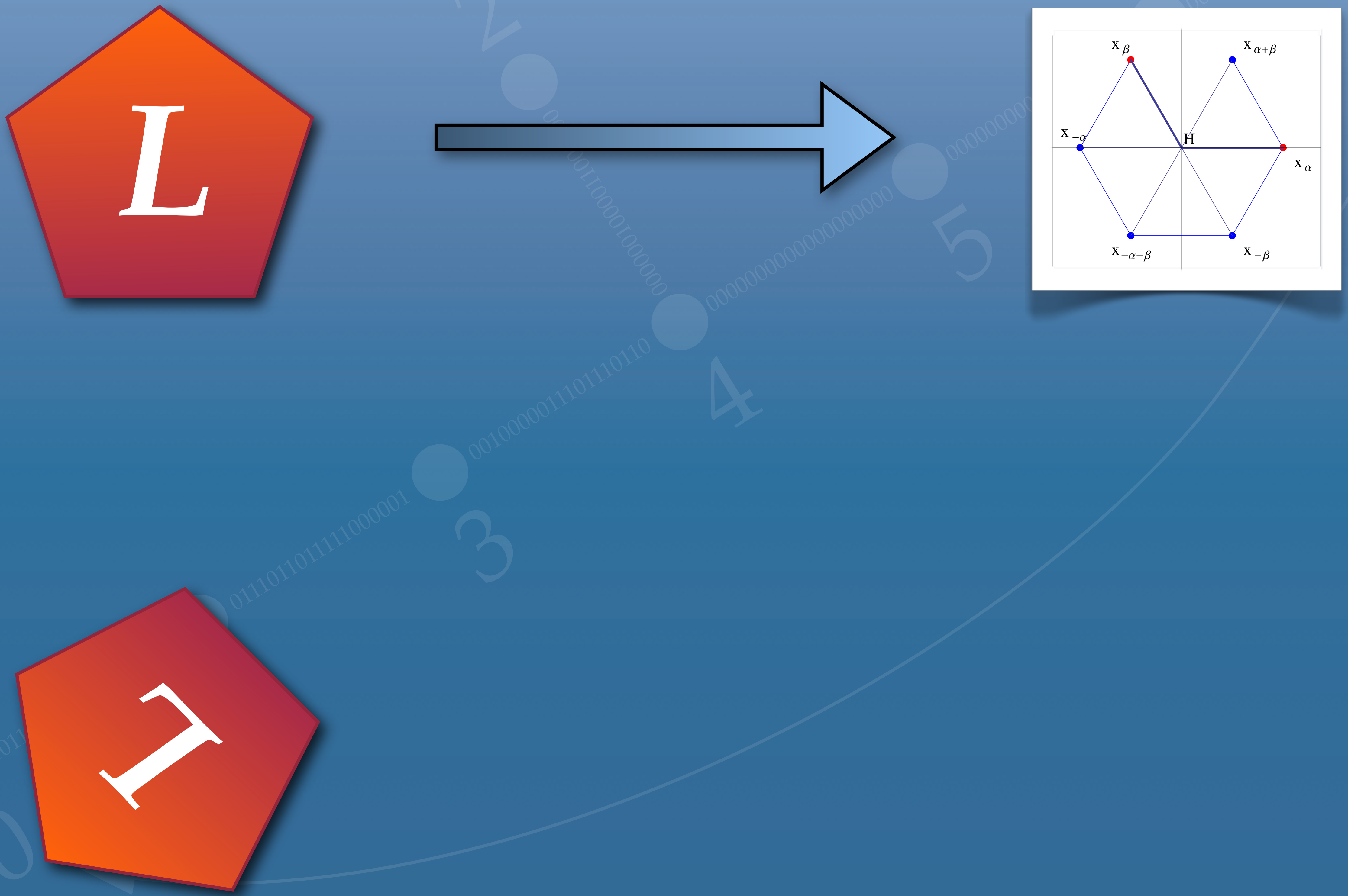

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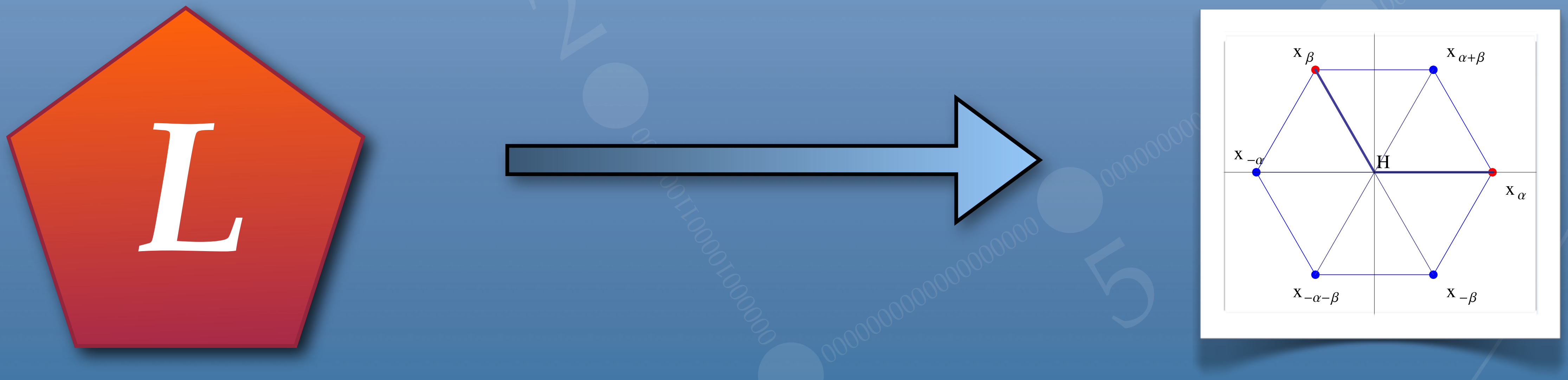

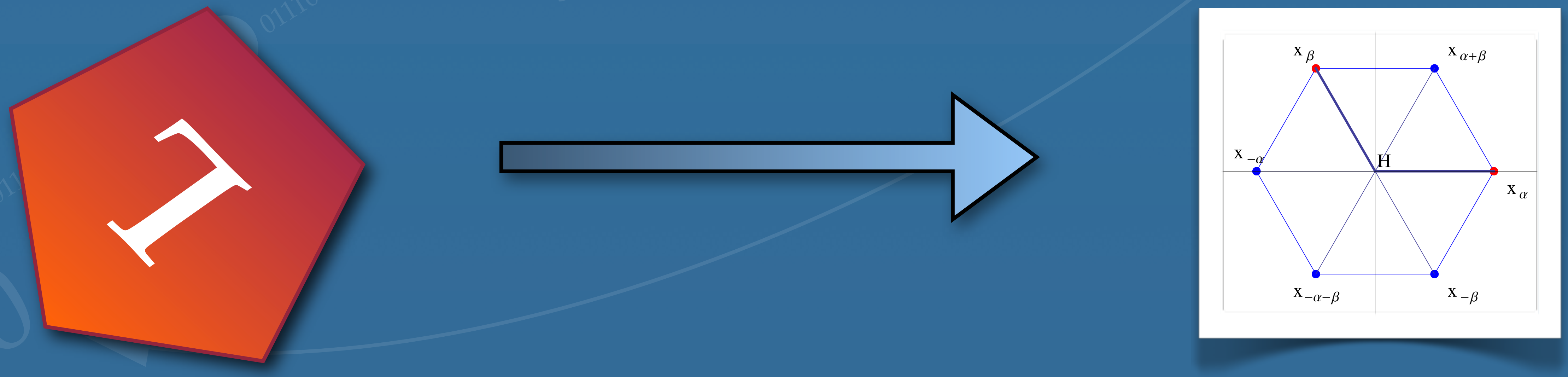

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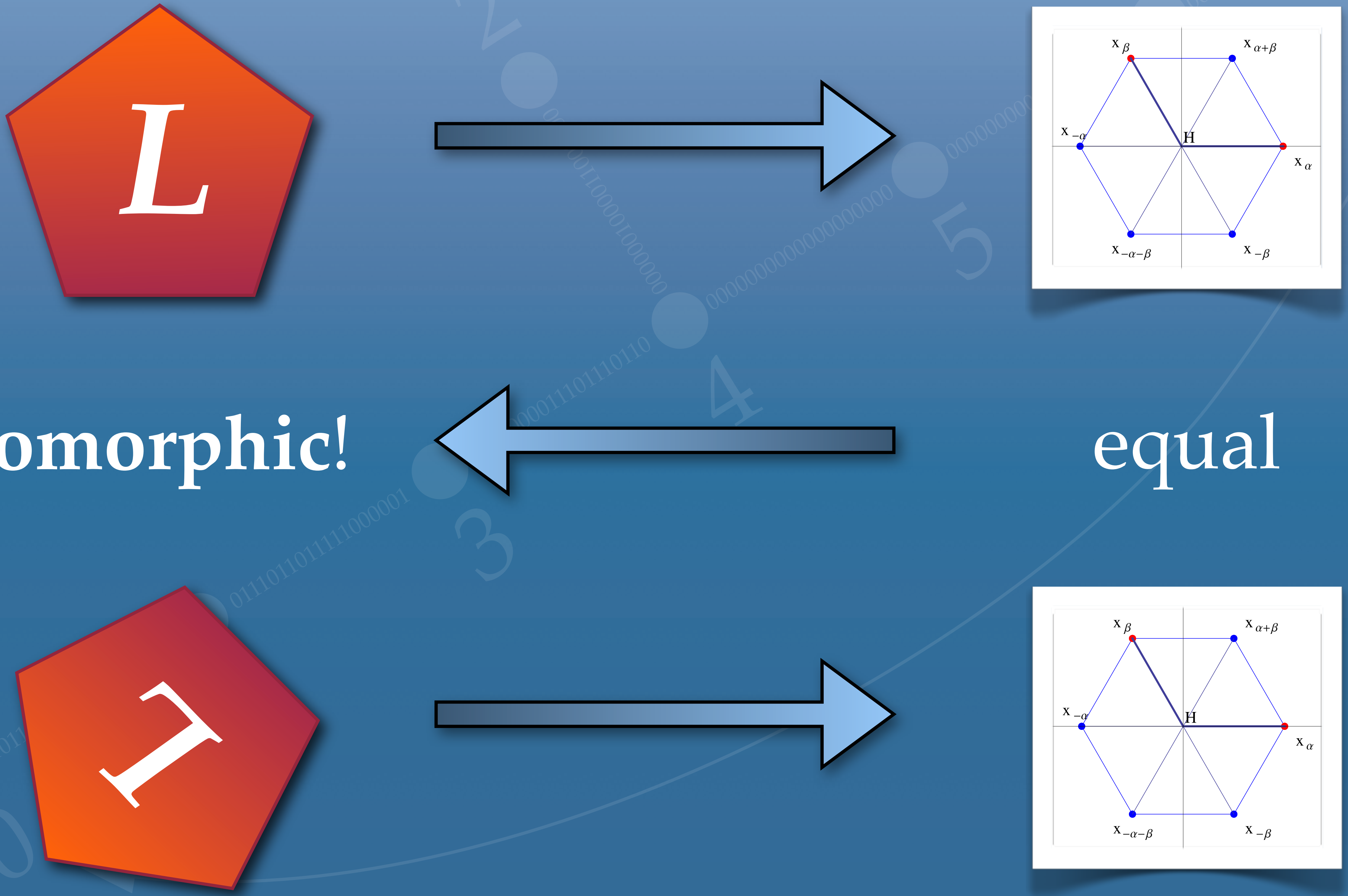

isomorphic!

equal

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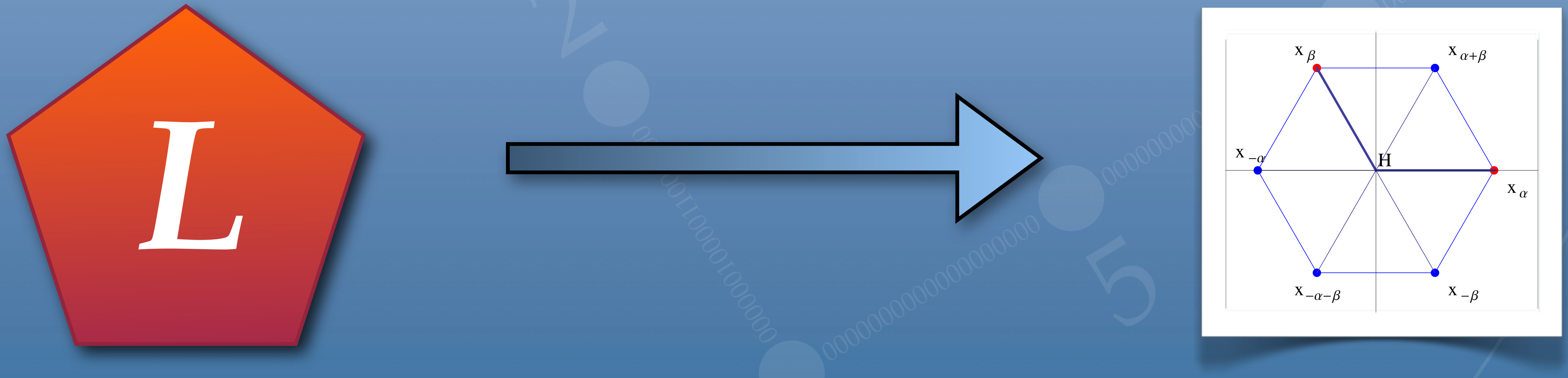

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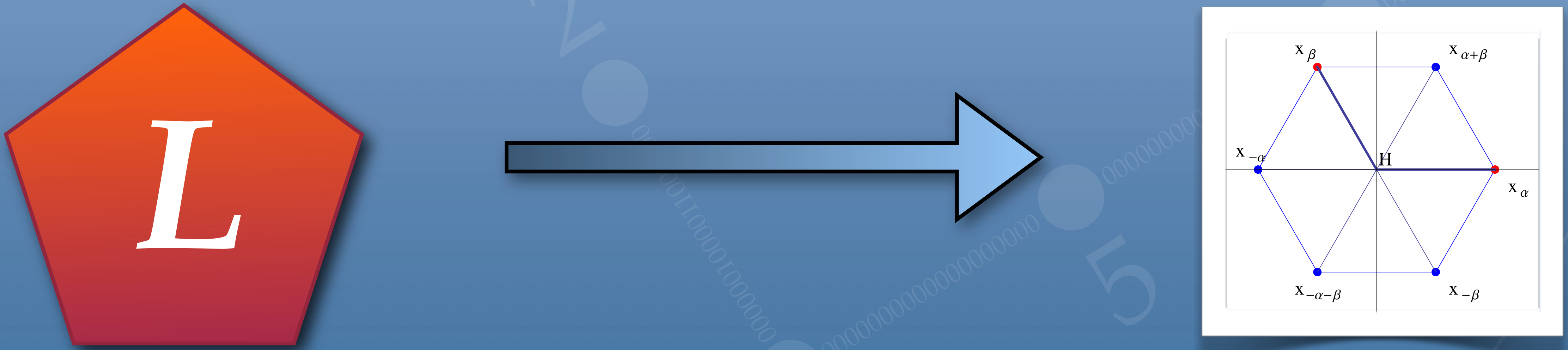

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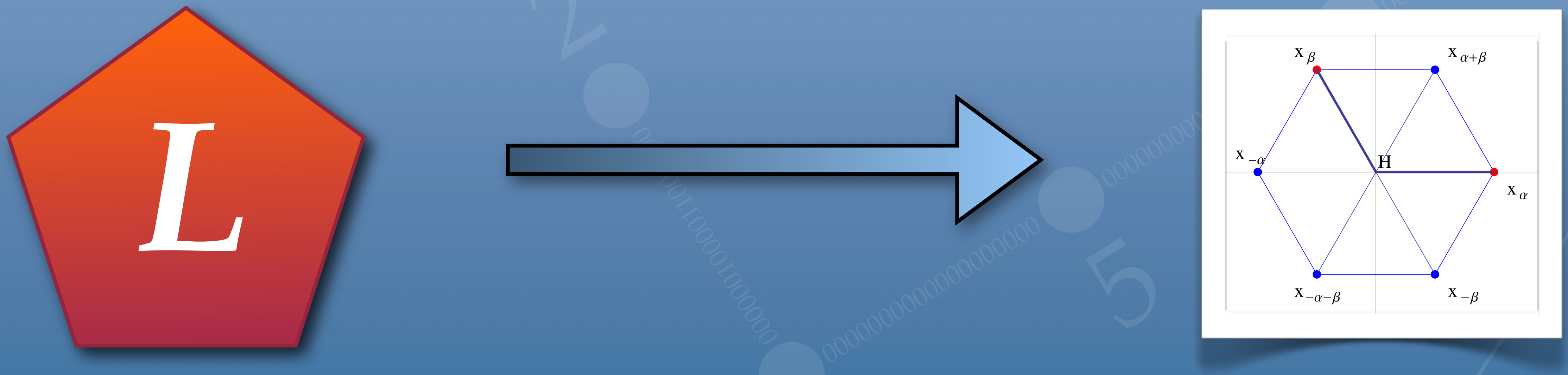

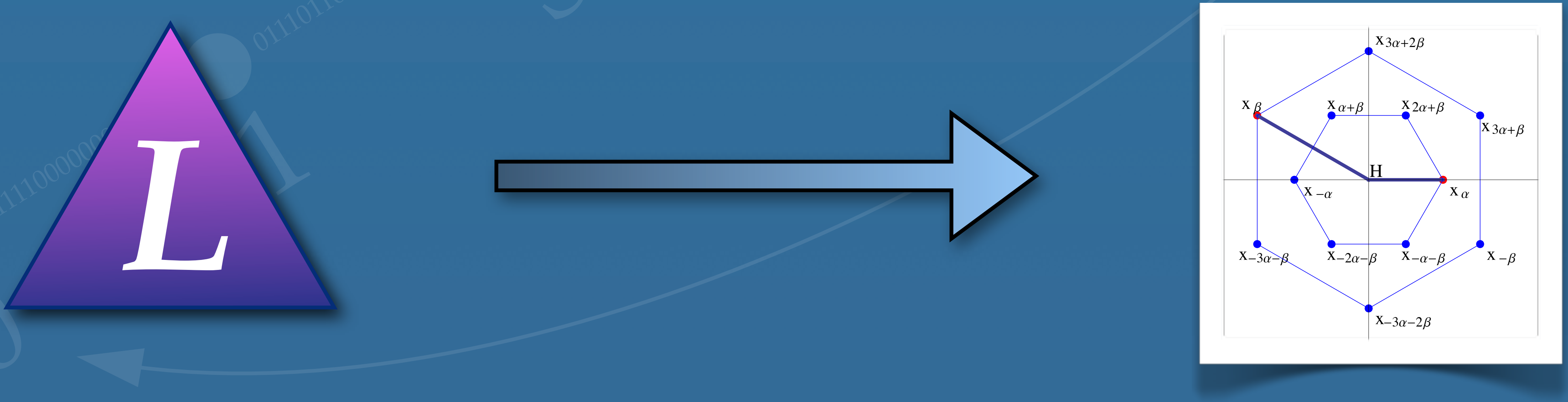

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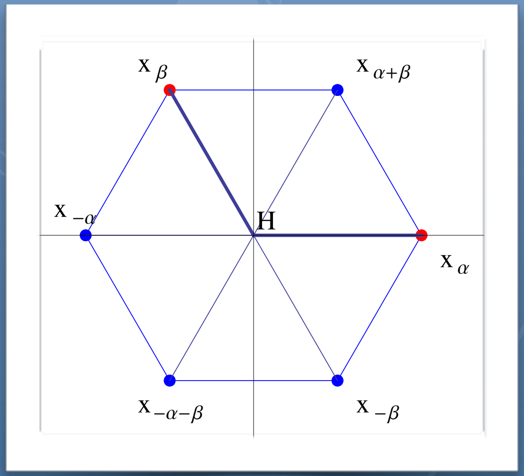
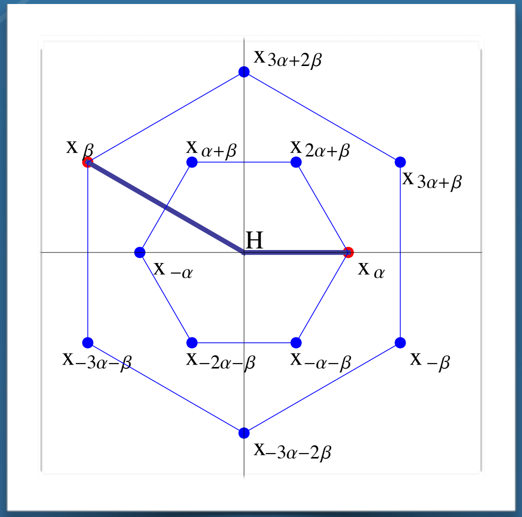


non-isomorphic!

not equal



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The Mission

- Given a Lie algebra (on a computer),
- We want to know which Lie algebra it is,
- So want to compute a Chevalley basis for it.

Root datum, field

Group of Lie type

Matrix Lie algebra

....

L

Chevalley Basis Algorithm

The Mission

Root datum, field

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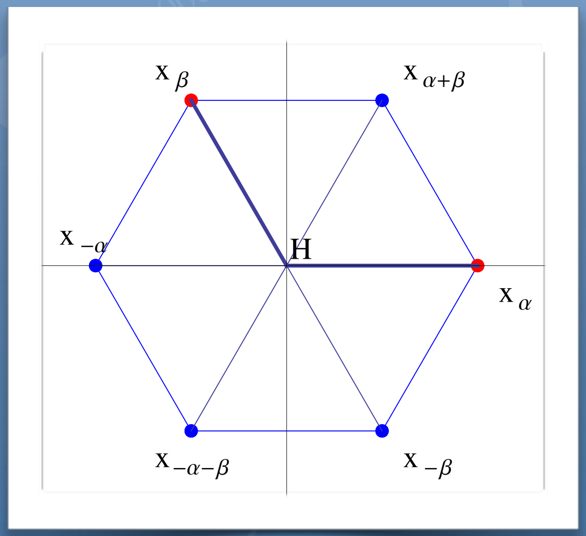
Chevalley Basis Algorithm



- Assume a *split Cartan subalgebra H* is given (separate problem; Cohen/Murray, indep. Ryba)
- Assume *R* is given (easy to find)

The Mission

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L

Chevalley Basis Algorithm

Algorithms:

- $\mathrm{char}(\mathbb{F}) = 0, \geq 5:$ De Graaf, Murray (implemented in GAP, Magma)
- $\mathrm{char}(\mathbb{F}) = 2, 3:$ Cohen, R. (implemented in Magma)

Computing Chevalley bases

- Given this Lie algebra L over $\mathbb{Q}$:

$[\cdot,\cdot]$	x_1	x_2	x_3
x_1	0	$-148x_1+158x_2+48x_3$	$290x_1-168x_2-158x_3$
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- and split Cartan subalgebra $H=\mathbb{Q}(-6x_1+5x_2+3x_3)$

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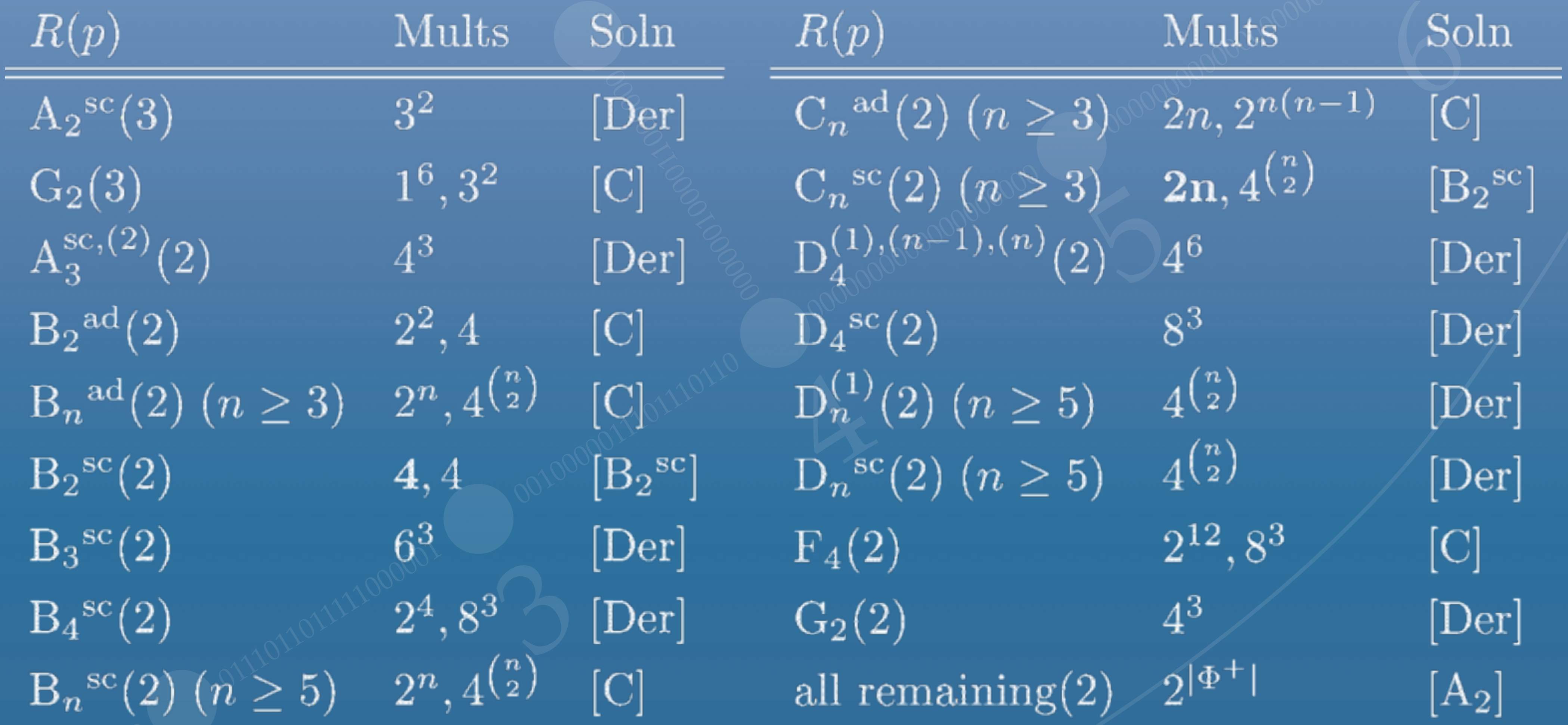
Computing Chevalley bases

- Use *Cartan integers* to identify $\mathbb{Q}x_{\alpha}$,

- Remedied by:
  - A_n, D_n, E_6, E_7, E_8 : Use $[x_{-\alpha}, x_{\beta}]$ and $[x_{-\alpha}, [x_{-\alpha}, x_{\beta}]]$ to determine $\langle \alpha, \beta^{\vee} \rangle$,
  - B_n, C_n, F_4, G_2 : Use ideals of type D_n, D_n, D_4, A_2, resp.

Computing Chevalley bases

- Diagonalise L wrt H

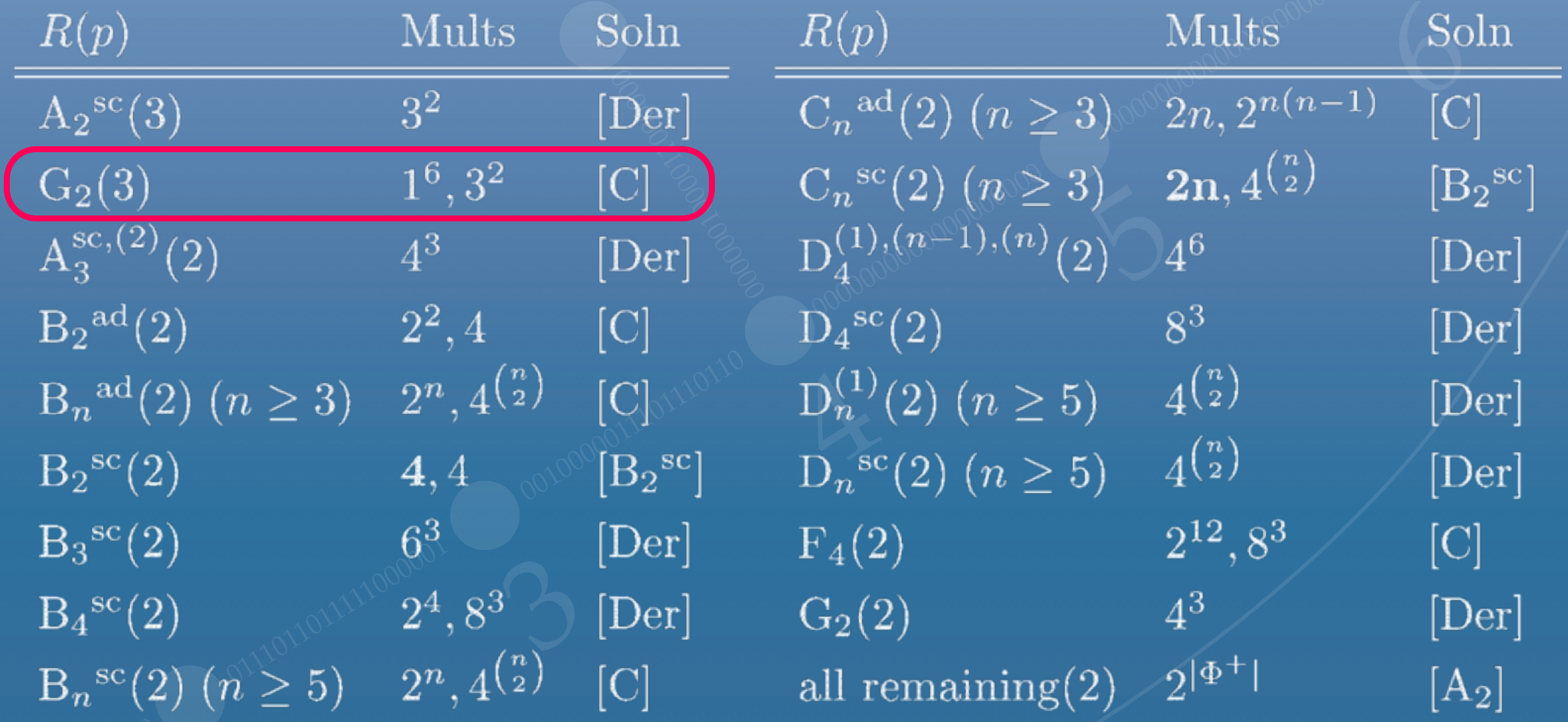


$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$\mathrm{A}_2{}^{\mathrm{sc}}(3)$	3^2	[Der]	$\mathrm{C}_n{}^{\mathrm{ad}}(2)\ (n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$\mathrm{G}_2(3)$	$1^6, 3^2$	[C]	$\mathrm{C}_n{}^{\mathrm{sc}}(2)\ (n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	[$\mathrm{B}_2{}^{\mathrm{sc}}$]
$\mathrm{A}_3^{\mathrm{sc},(2)}(2)$	4^3	[Der]	$\mathrm{D}_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$\mathrm{B}_2{}^{\mathrm{ad}}(2)$	$2^2, 4$	[C]	$\mathrm{D}_4{}^{\mathrm{sc}}(2)$	8^3	[Der]
$\mathrm{B}_n{}^{\mathrm{ad}}(2)\ (n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$\mathrm{D}_n^{(1)}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B}_2{}^{\mathrm{sc}}(2)$	$\mathbf{4}, 4$	[$\mathrm{B}_2{}^{\mathrm{sc}}$]	$\mathrm{D}_n{}^{\mathrm{sc}}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B}_3{}^{\mathrm{sc}}(2)$	6^3	[Der]	$\mathrm{F}_4(2)$	$2^{12}, 8^3$	[C]
$\mathrm{B}_4{}^{\mathrm{sc}}(2)$	$2^4, 8^3$	[Der]	$\mathrm{G}_2(2)$	4^3	[Der]
$\mathrm{B}_n{}^{\mathrm{sc}}(2)\ (n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	[A_2]

TABLE 1. Multidimensional root spaces

Computing Chevalley bases

- Diagonalise L wrt H



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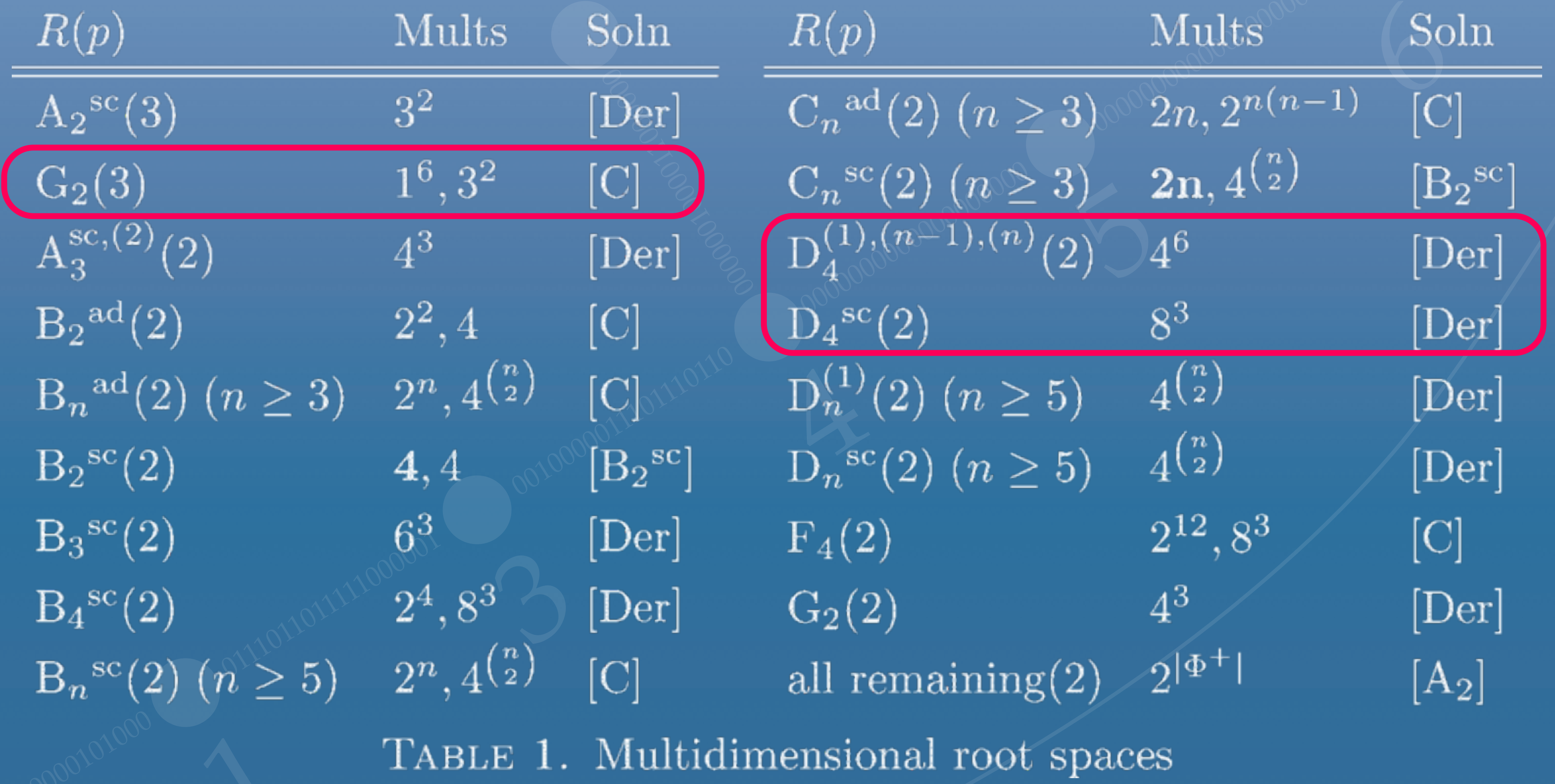

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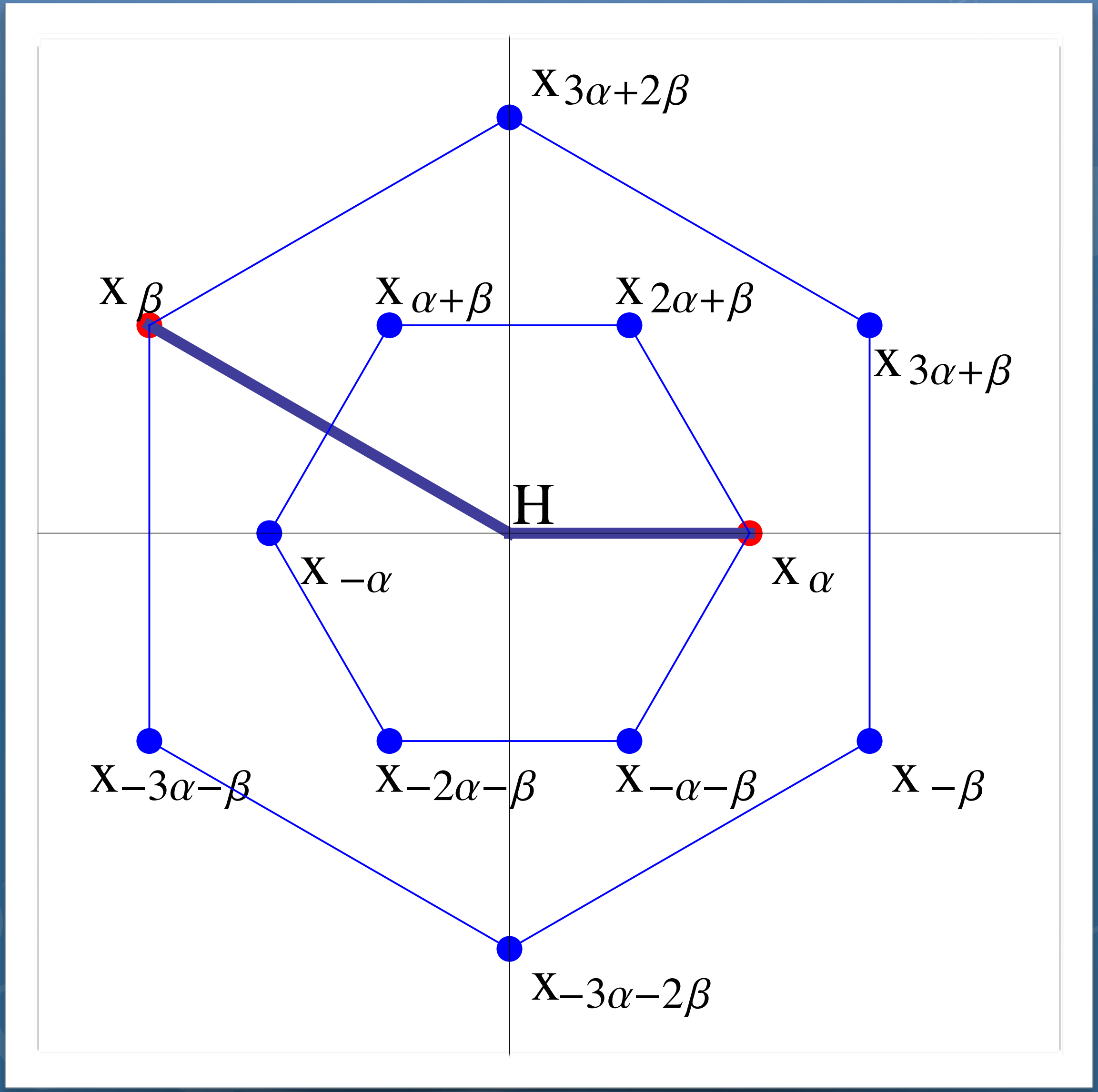
Computing Chevalley bases

- Diagonalise L wrt H: The *centraliser* method

G_2, char. 3



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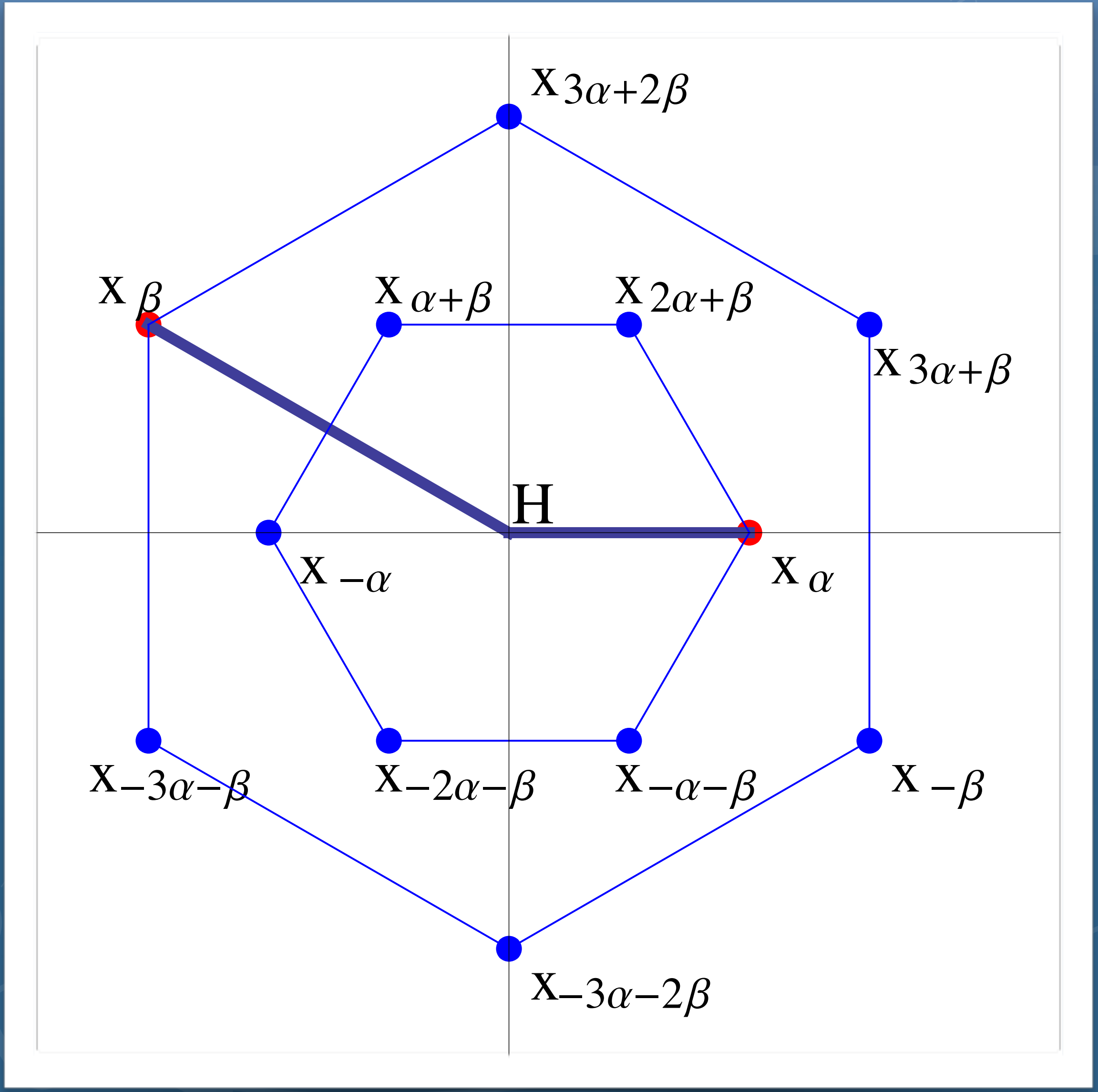

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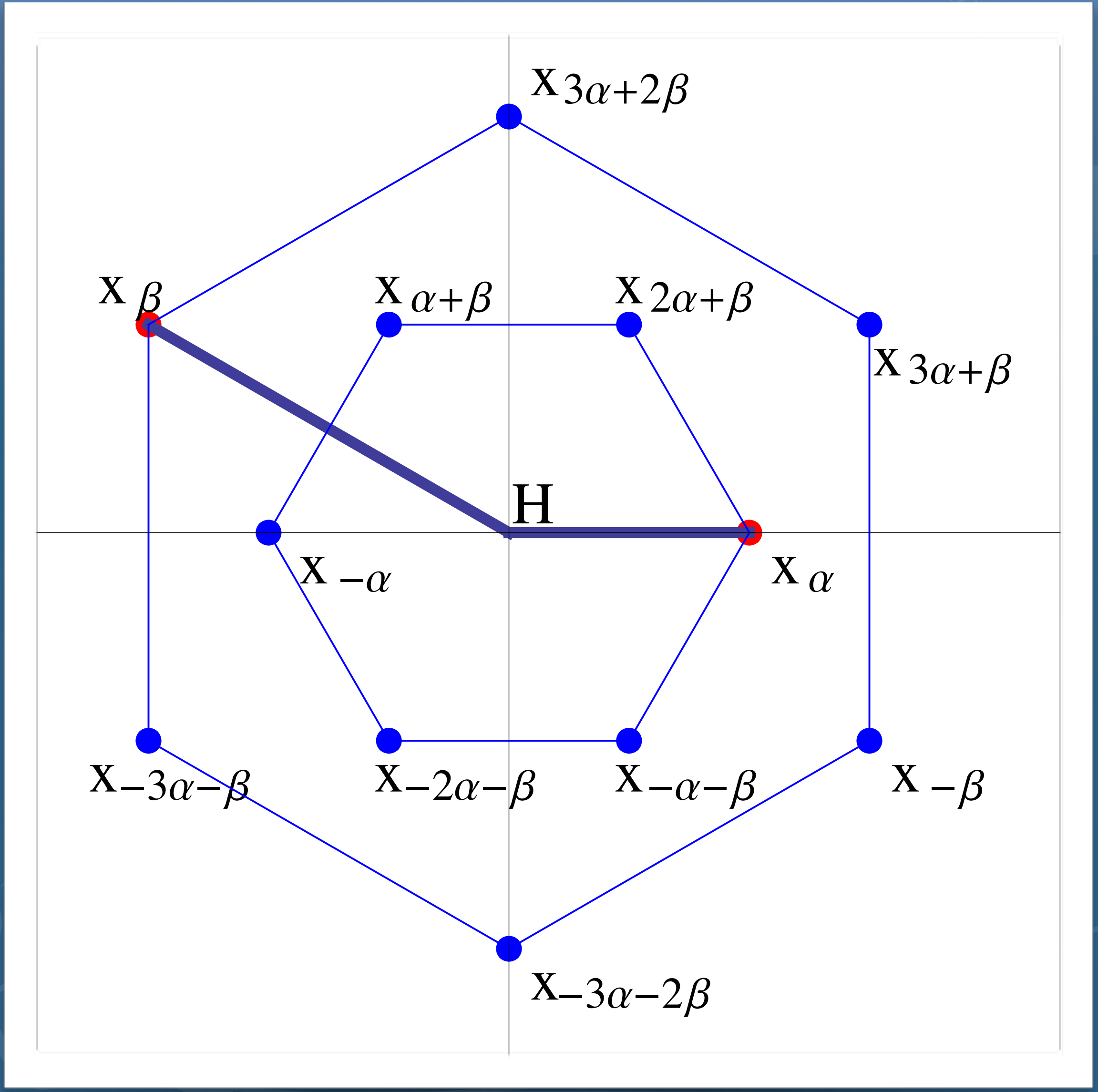

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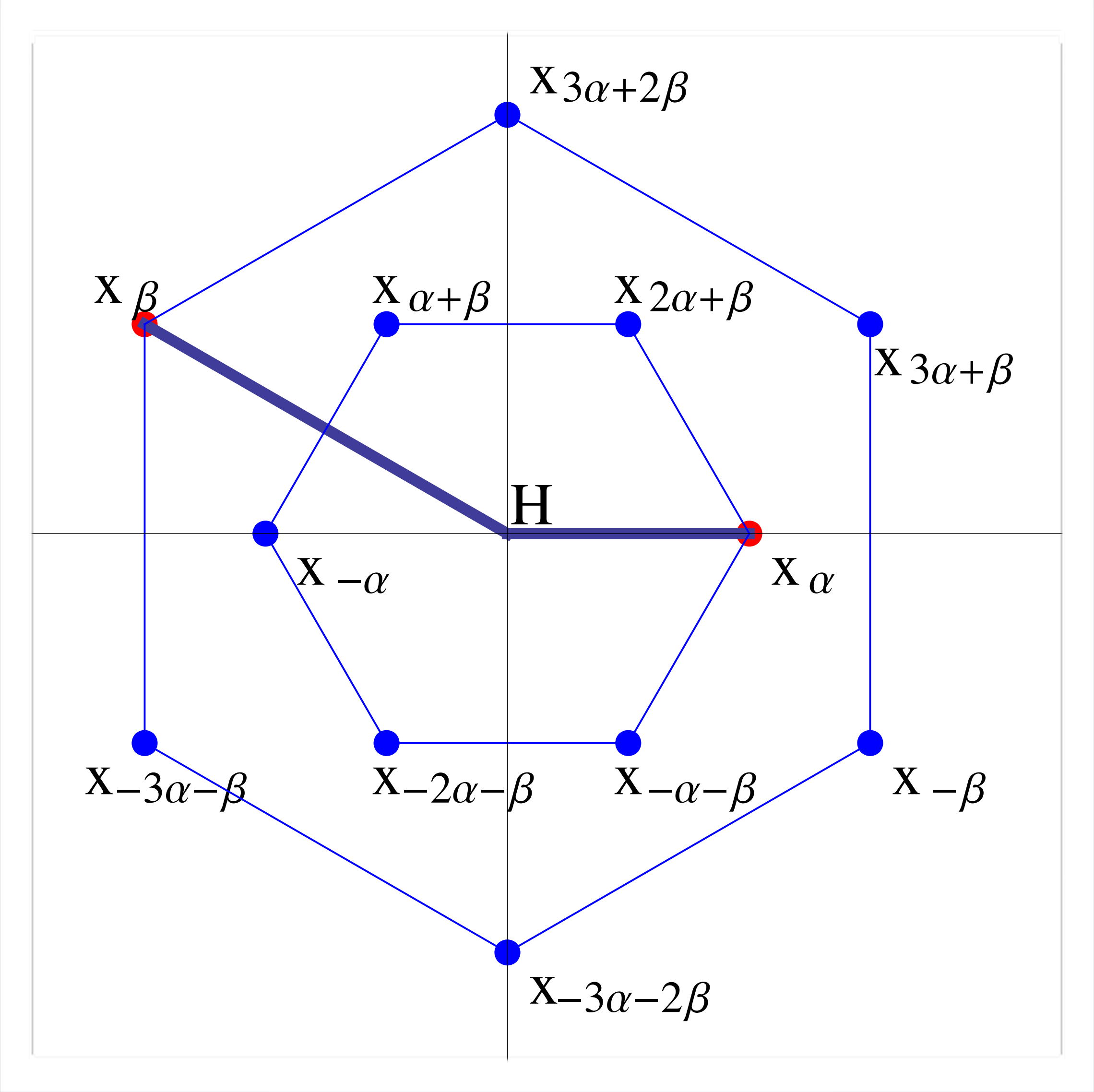
$[\cdot,\cdot]$	x_α	$x_{-\alpha}$
x_β	$-x_{\alpha+\beta}$	0
$x_{-3\alpha-2\beta}$	0	0
$x_{3\alpha+\beta}$	0	$-x_{2\alpha+\beta}$

G_2, char. 3



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1	26	1
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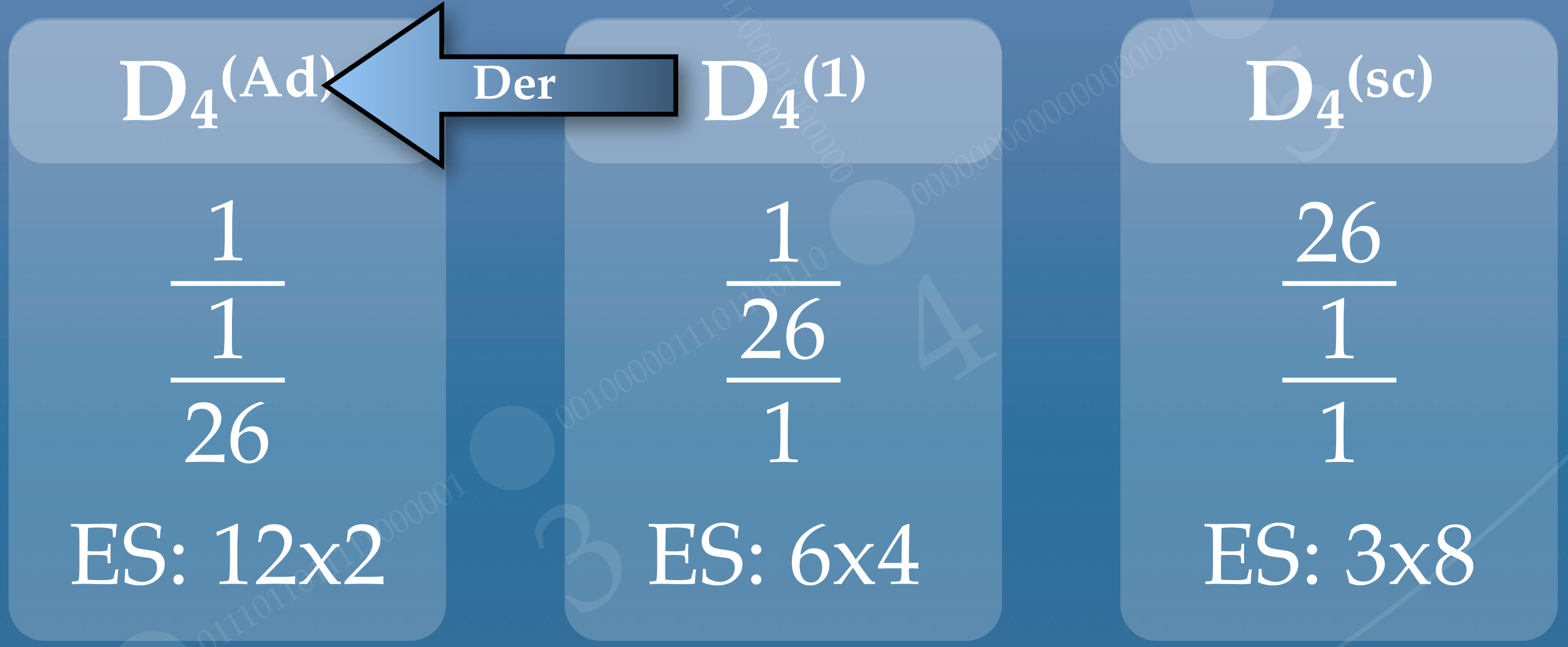

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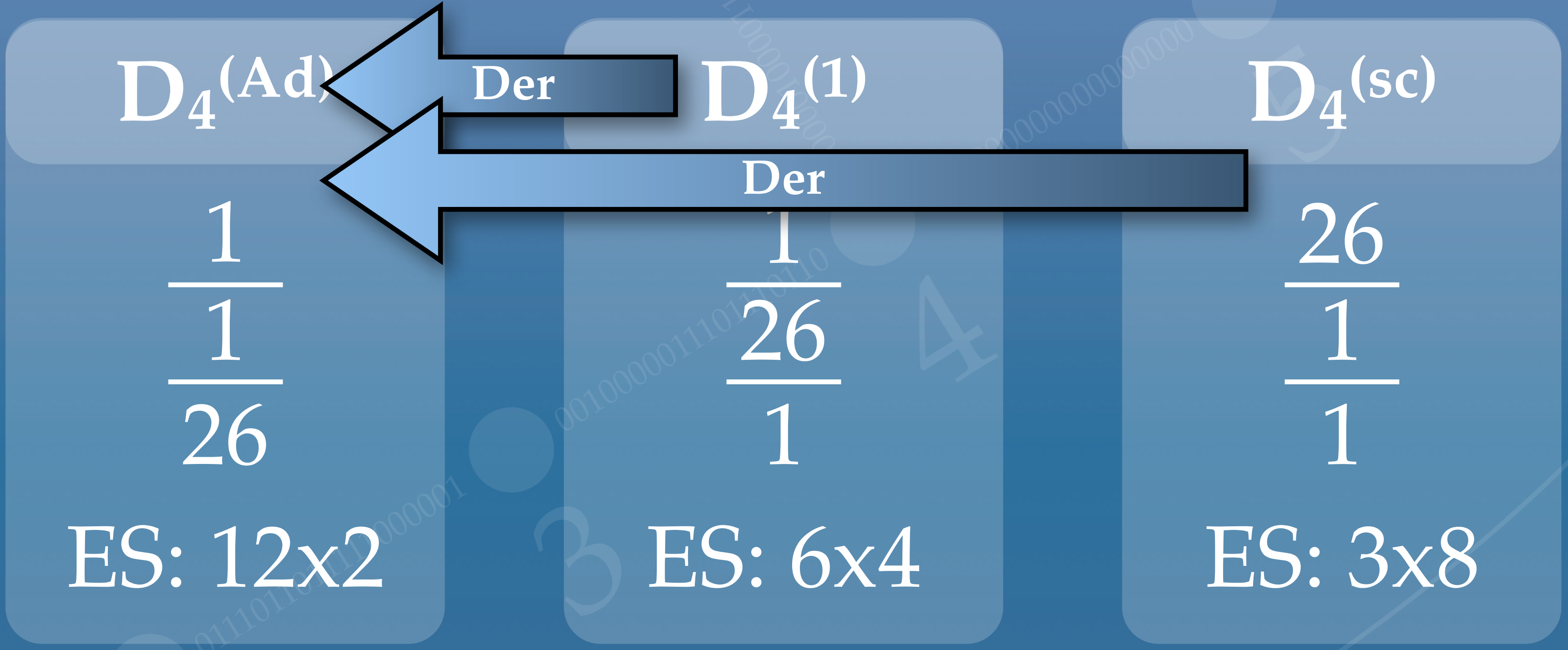

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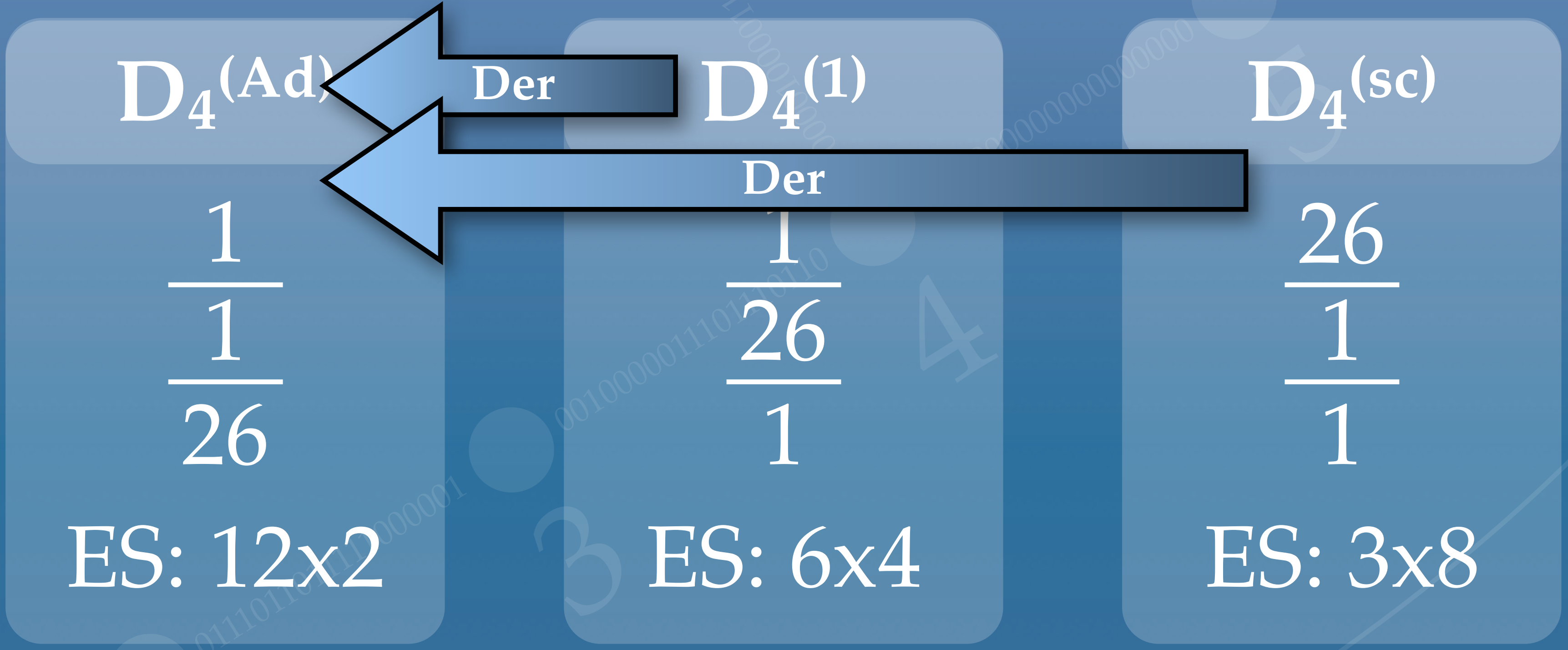

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- Find root spaces in $D_4^{(1)}$ and $D_4^{(sc)}$ using the eigenspaces of $\mathrm{Der}(L) = D_4^{(Ad)}$

Computing Chevalley bases

- Normally:
  - Diagonalise L wrt H,
  - Use *Cartan integers* to identify $\mathbb{Q}x_\alpha$,
  - Solve easy linear equations.

Computing Chevalley bases

- In char. 2, 3:
  - Diagonalise L wrt H,
  - Go through pains to get root spaces $\mathbb{Q}x_\alpha$,
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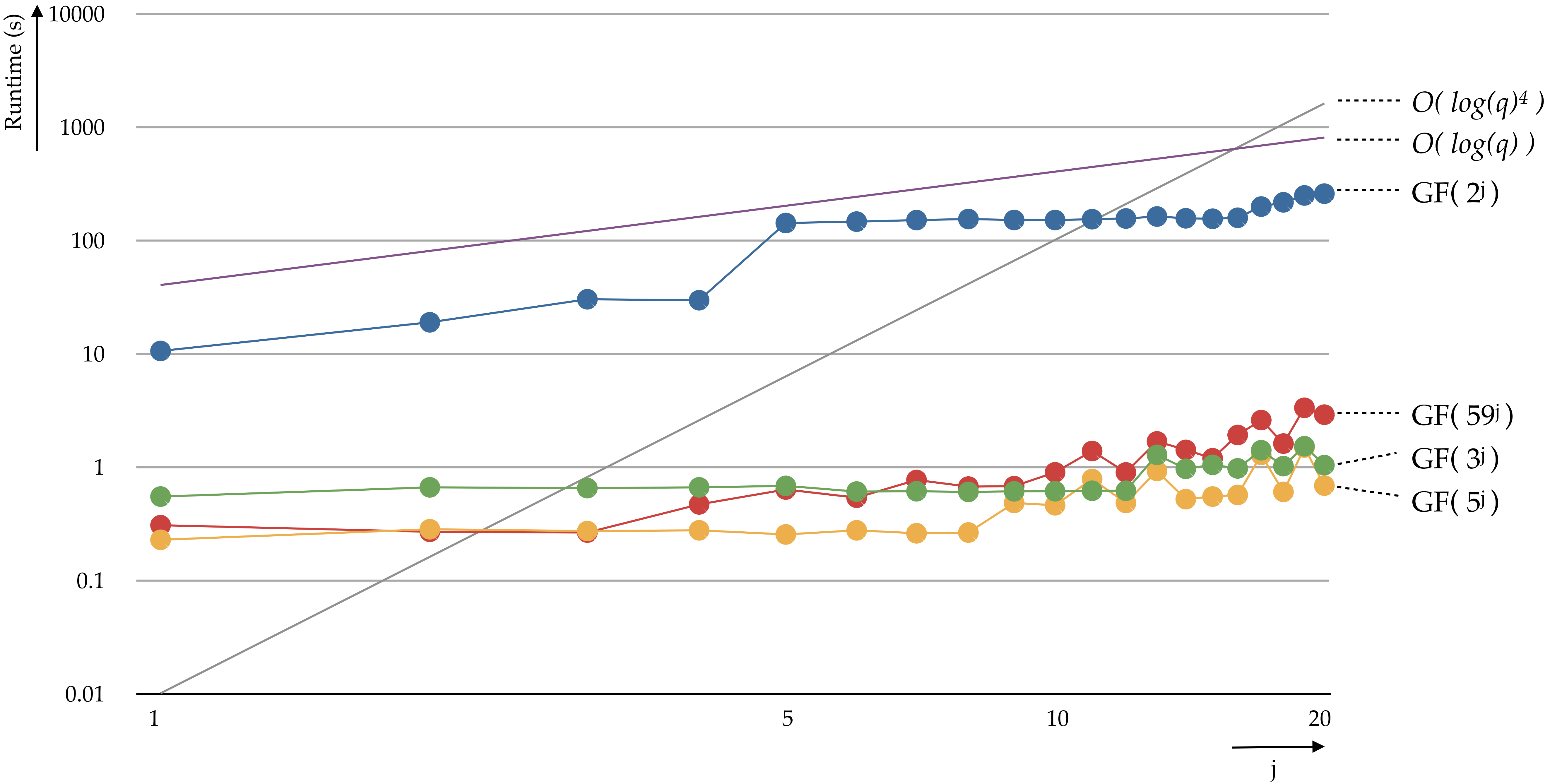
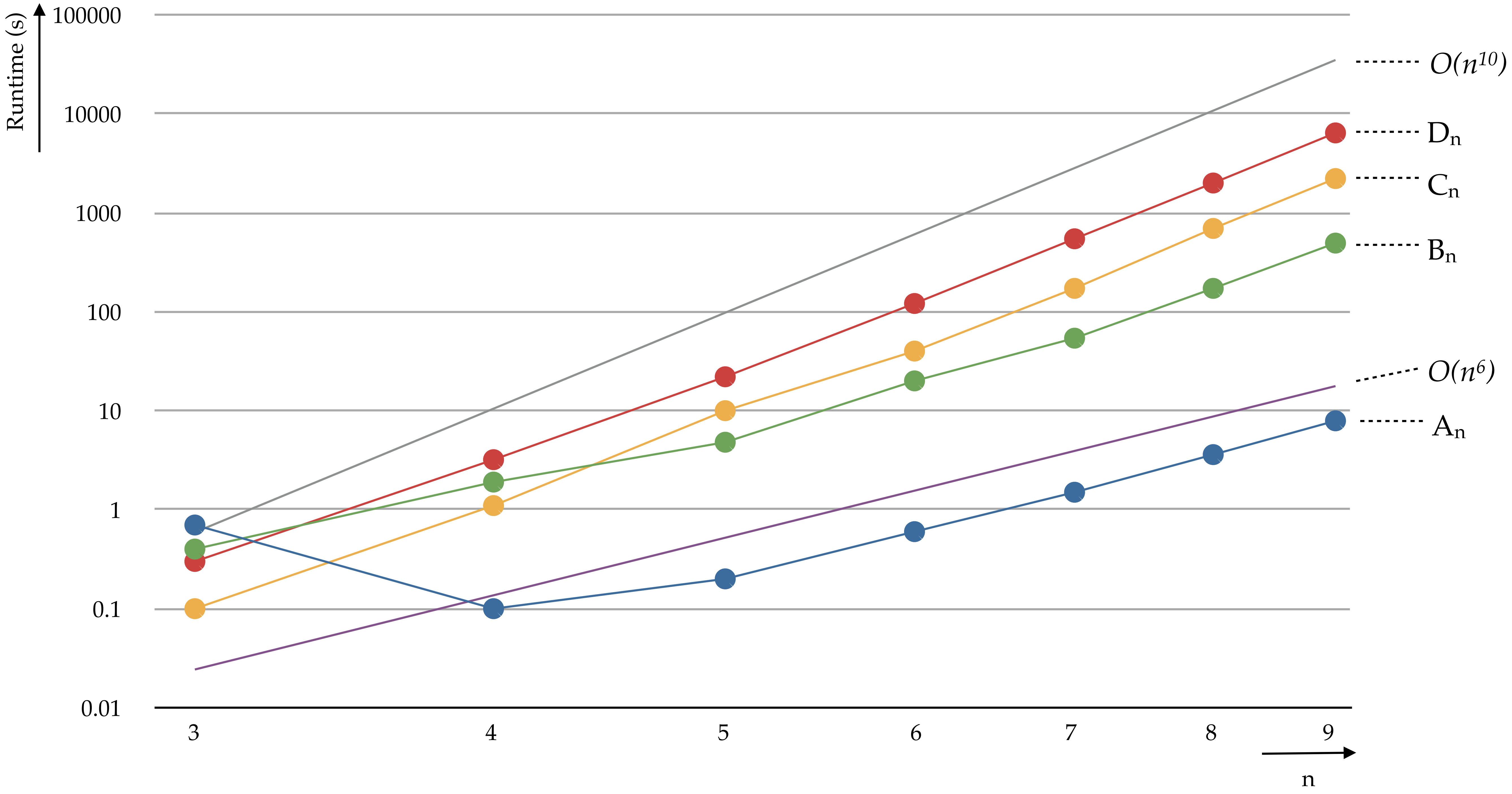
Runtimes



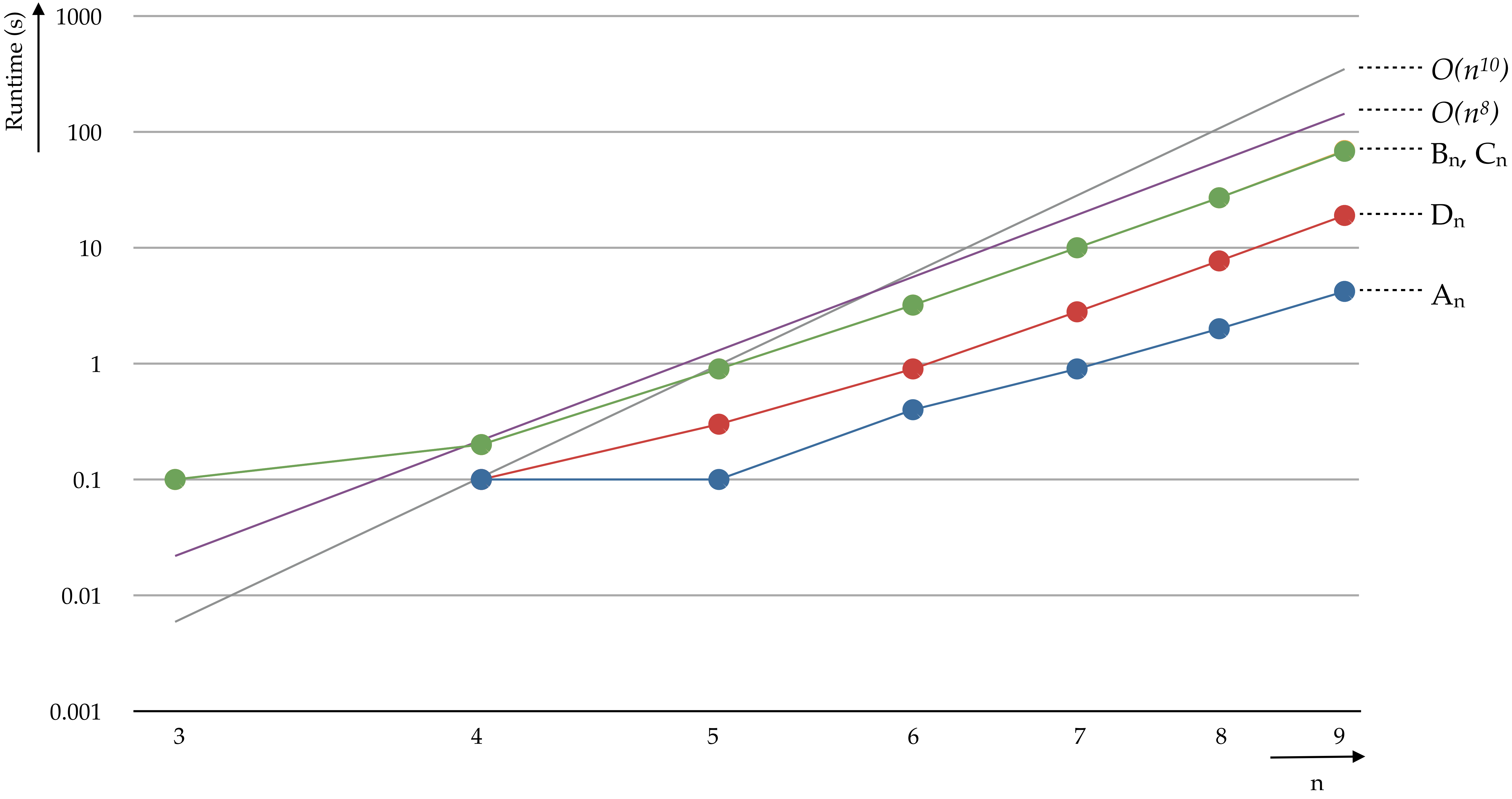
By field

Runtimes



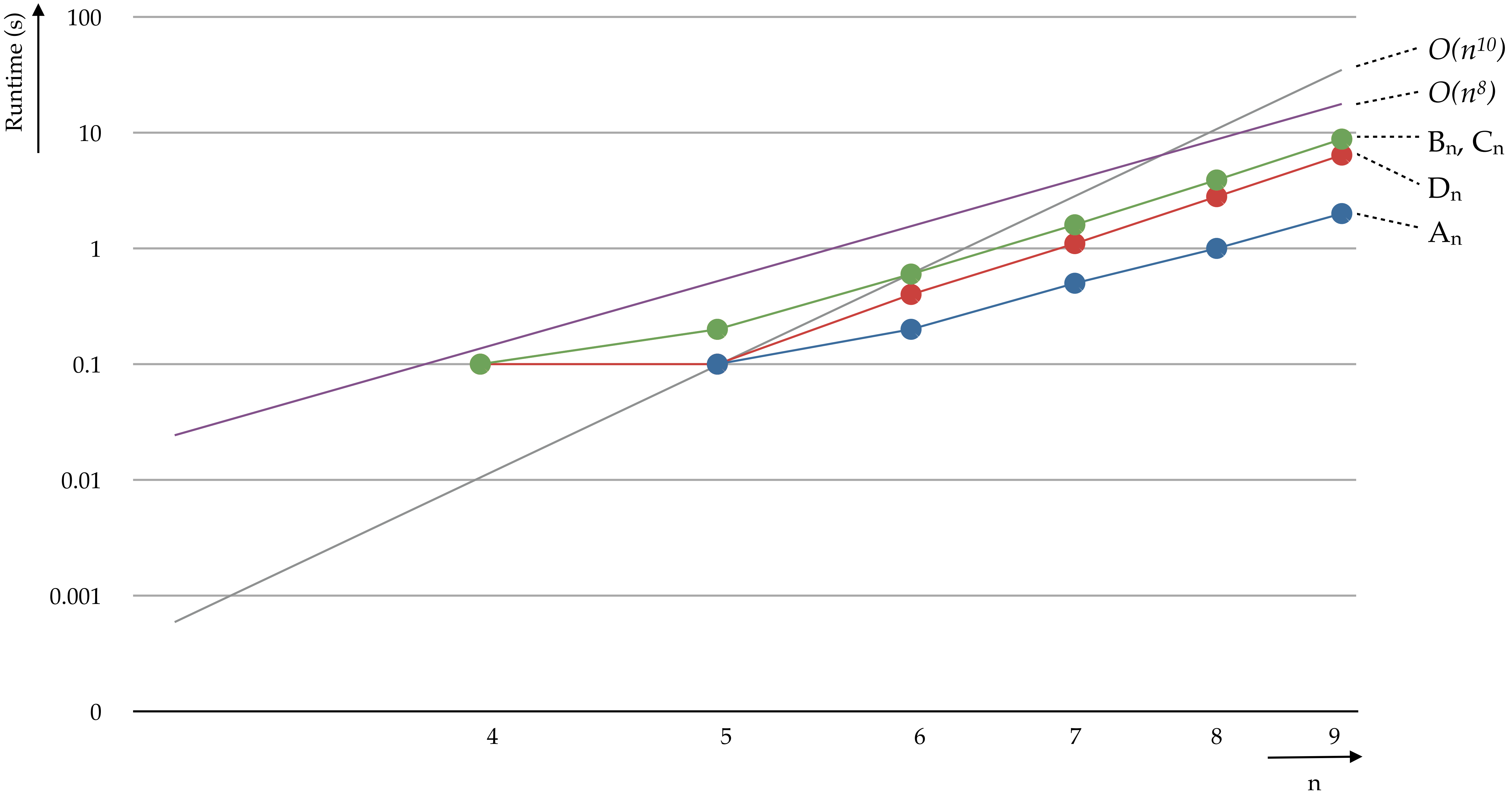
GF(2^6)

Runtimes



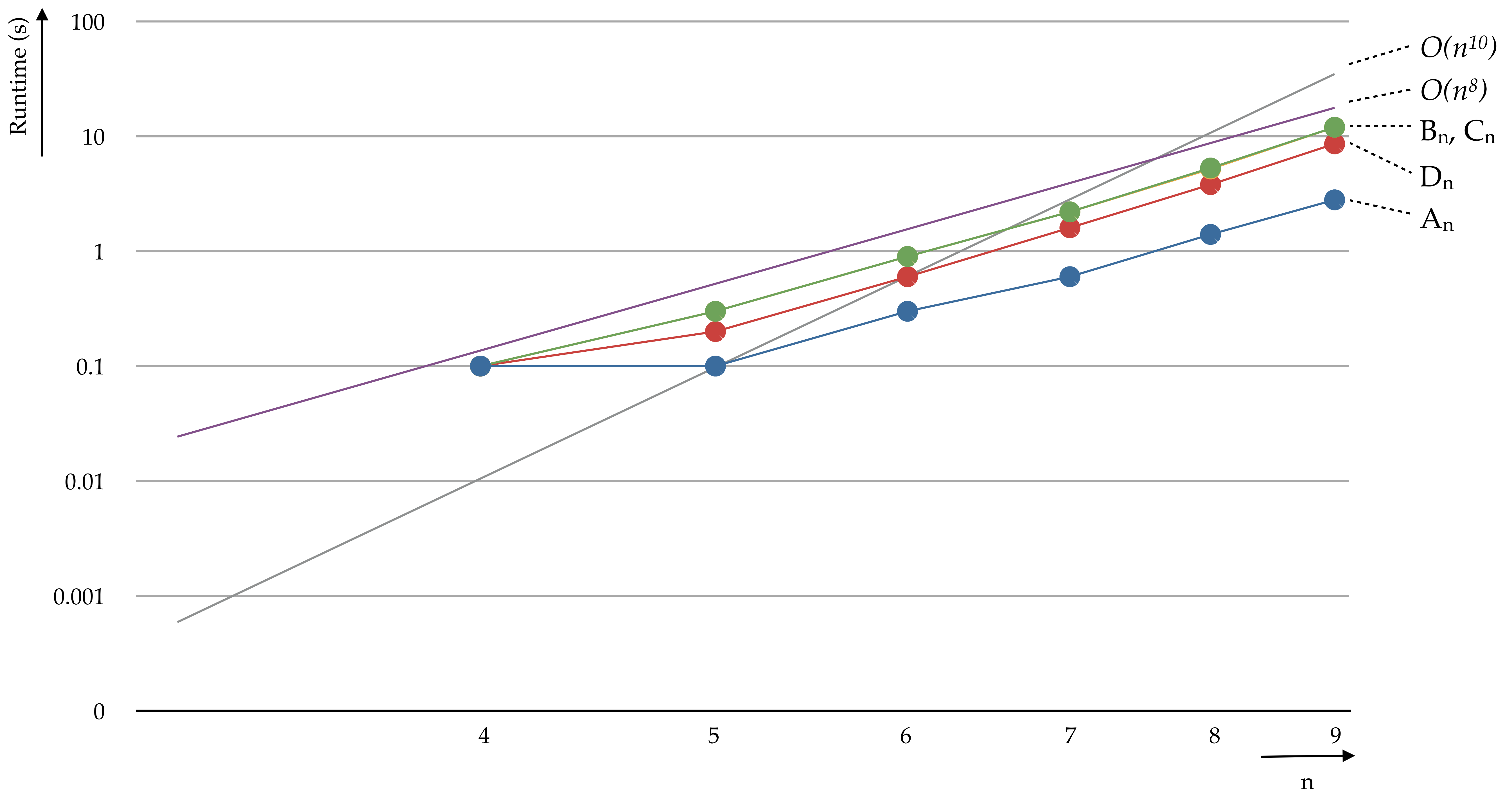
GF(3^3)

Runtimes



GF(17)

Runtimes



Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- What does 537 mean?

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Definition (*Extremal element*)

An element x of a Lie algebra L is called *extremal* if, for all $y \in L$: $[x,[x,y]] \in \mathbb{F}x$.

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For the remainder
$\mathrm{char}(\mathbb{F}) \neq 2$

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$[\cdot,\cdot]$	x	y	$[x,y]$
x	0	$[x,y]$	αx
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$$[y,[x,[x,y]]]$$

$$= \quad [x,[y,[x,y]]] + [[x,y],[x,y]]$$

$$-$$

If $\alpha \neq 0$

then $L \cong A_1$!

Facts about Extremal Elements

Theorems (ZK1991 / CSUW2001)

If L is a Lie algebra generated by extr. elts. then

- L has a basis B consisting of extremal elements,
- There exists a bilinear symmetric associative f such that $[x, [x, y]] = f(x, y)x$ (for all x, y),
- The B above is independent from the choice of f,
- Minimal numbers of extremal generators:

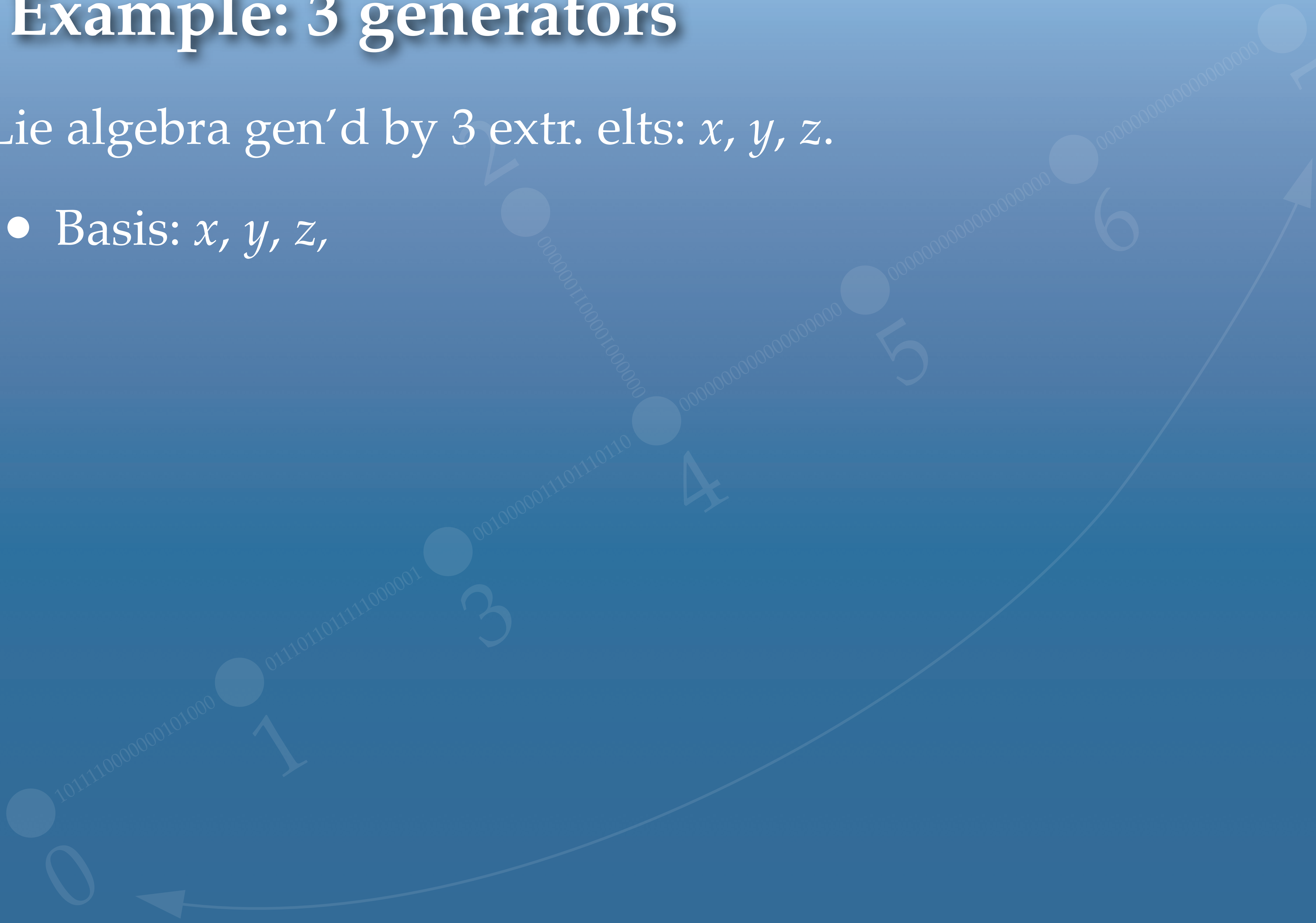
type	num. gens	cond.
A_n	$n+1$	$n \geq 1$
B_n	$n+1$	$n \geq 3$
C_n	$2n$	$n \geq 2$
D_n	n	$n \geq 4$

type	num. gens	cond.
E_n	5	$n = 6, 7, 8$
F_4	5	
G_2	4	

Example: 3 generators

Lie algebra gen’d by 3 extr. elts: x, y, z.

- Basis: x, y, z,



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Example: 3 generators

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- Multiplication: ....
- "Free" parameters: $f(x,y)$, $f(x,z)$, $f(y,z)$, $f(x,[y,z])$.

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- Chevalley basis! W.r.t. root system of type A_2 $(\mathbb{F} = \mathrm{GF}(5))$

$$\begin{aligned} x_\alpha &= -2x \\ x_\beta &= y - 2[x,y] - 2[y,z] - [y,[x,z]] \\ x_{\alpha+\beta} &= -2x + [x,y] - [x,z] - [x,[y,z]] \\ x_{-\alpha} &= y \\ x_{-\beta} &= -x + 2[x,y] + 2[x,z] - [x,[y,z]] \\ x_{-\alpha-\beta} &= y - 2[x,y] + 2[y,z] + 2[y,[x,z]] \end{aligned}$$

$$\begin{aligned} h_1 &= 2[x,y] \\ h_2 &= -x + y - 2[x,y] - 2[x,z] + [x,[y,z]] + [y,[x,z]] \end{aligned}$$

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$$\begin{aligned} f(x,y) &= 1, \\ f(x,z) &= 2, \\ f(y,z) &= 3, \\ f(x,[y,z]) &= 4. \end{aligned}$$

Example: 4 generators

Lie algebra L gen'd by 4 extr. elts: x, y, z, u.

- Basis: 28 elements: ...., $[u,[x,[y,[z,u]]]]$

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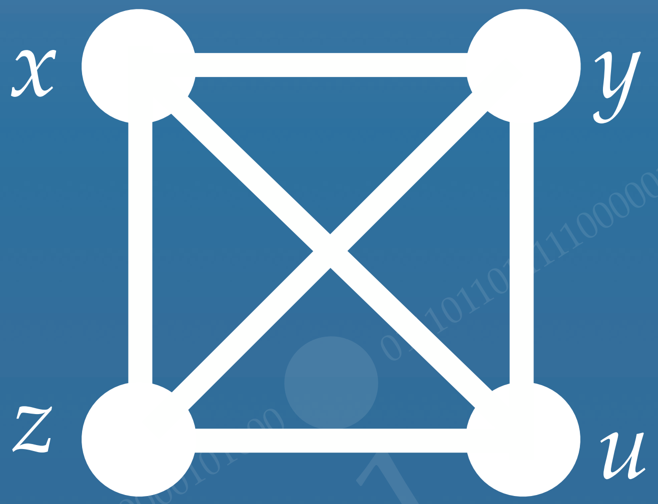

28-dim, D_4

Example: 4 generators

n	dim	type
2	3	A_1
3	8	A_2
4	28	D_4

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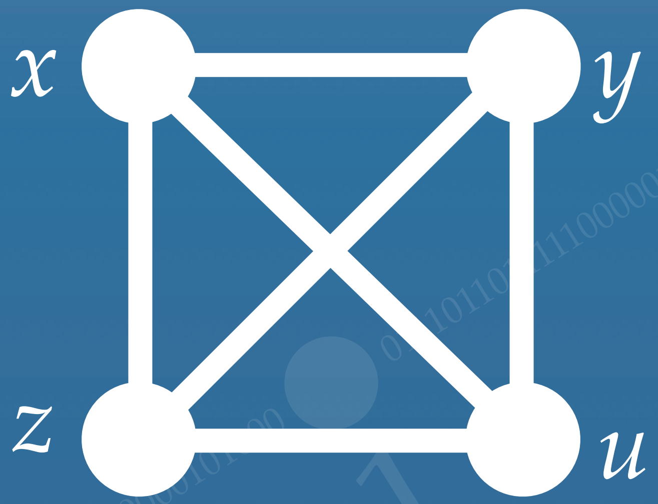

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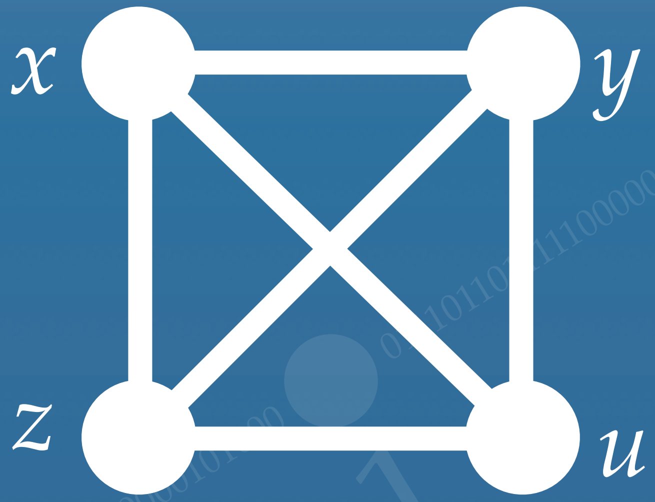

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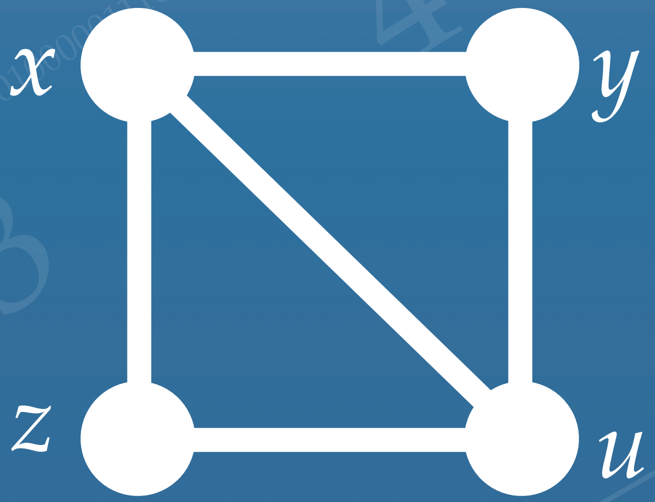

21-dim, B_3

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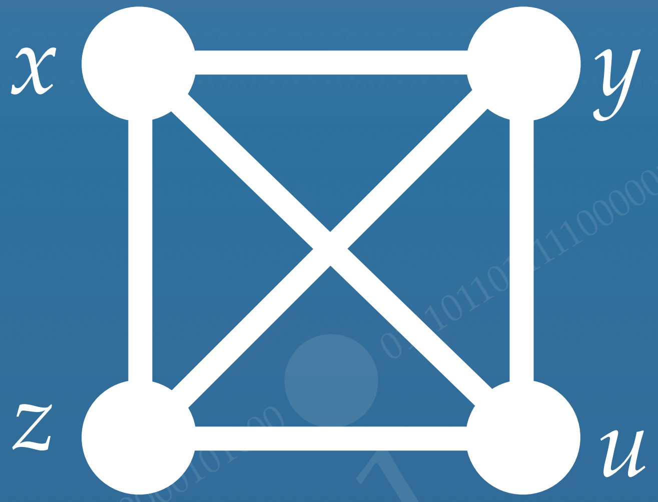

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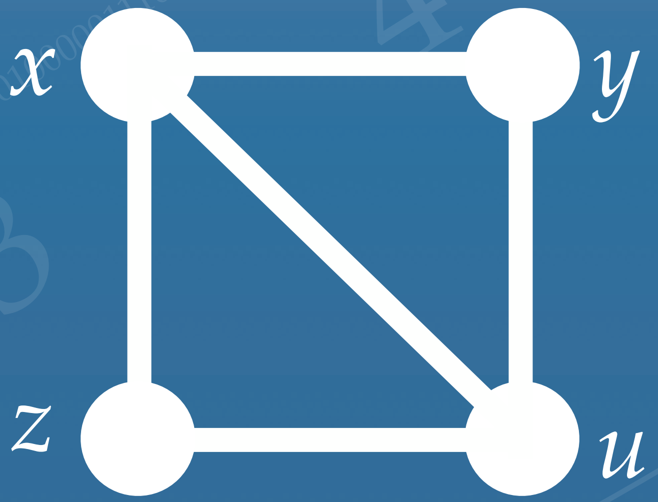

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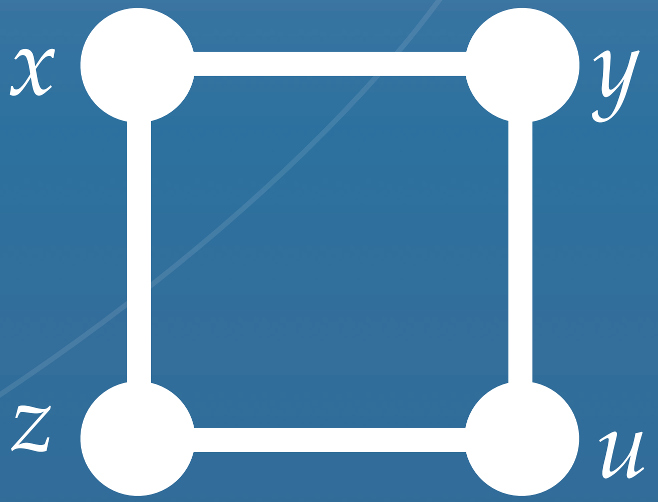
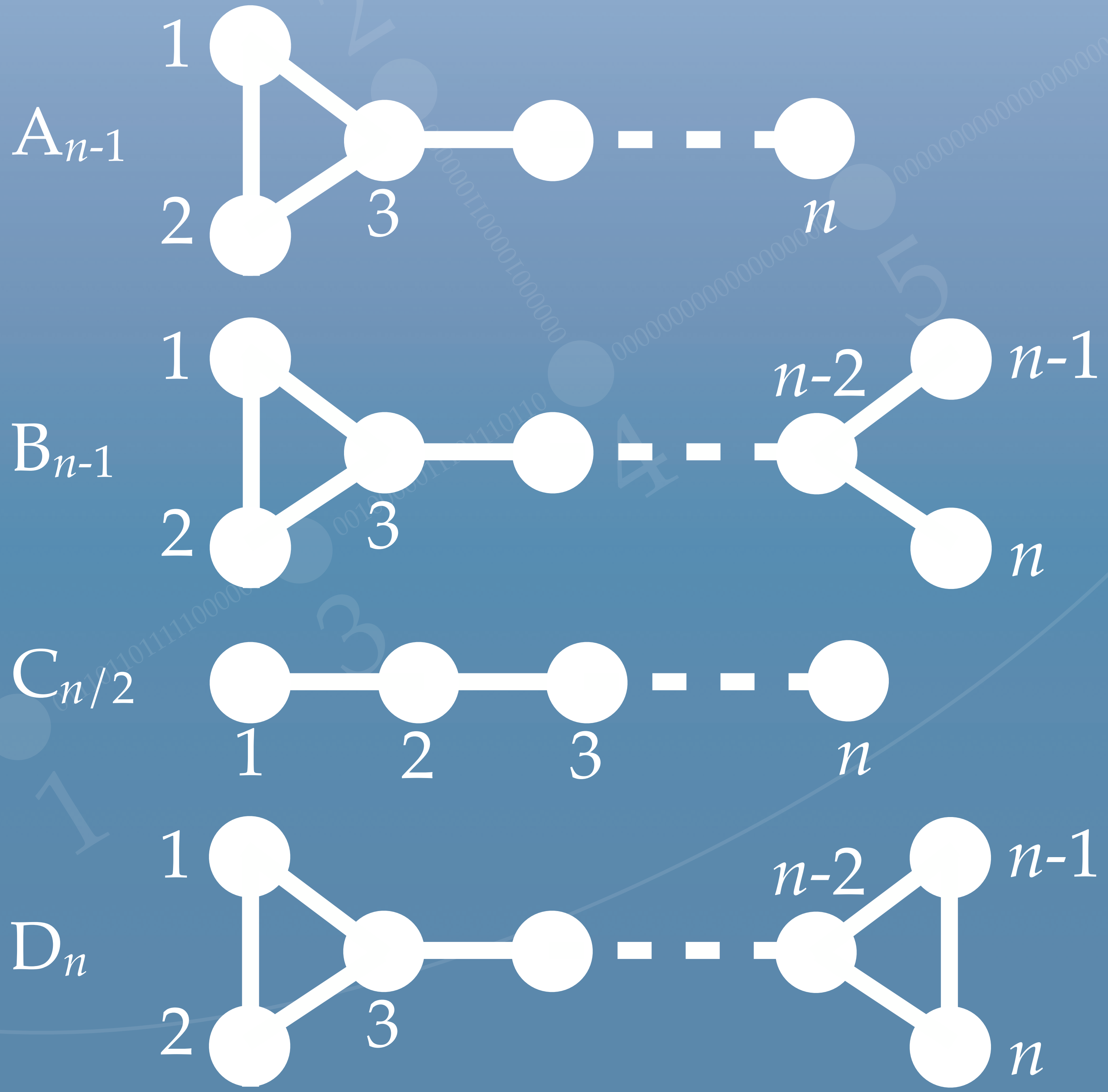


15-dim, A_3

The classical series

Theorem (in ‘t Panhuis, Postma, R., 2007)



Facts about Extremal Elements

Theorems (Draisma, in ‘t Panhuis, 2008)

If L is a Lie algebra generated by extr. elts. satisfying Γ, a finite graph, then:

- The choices for f such that L is of maximal dimension form an algebraic variety X,
- If Γ is a Dynkin diagram, then X is affine and all points in an open dense subset of X are Lie algebras isomorphic to a single fixed Lie algebra L',
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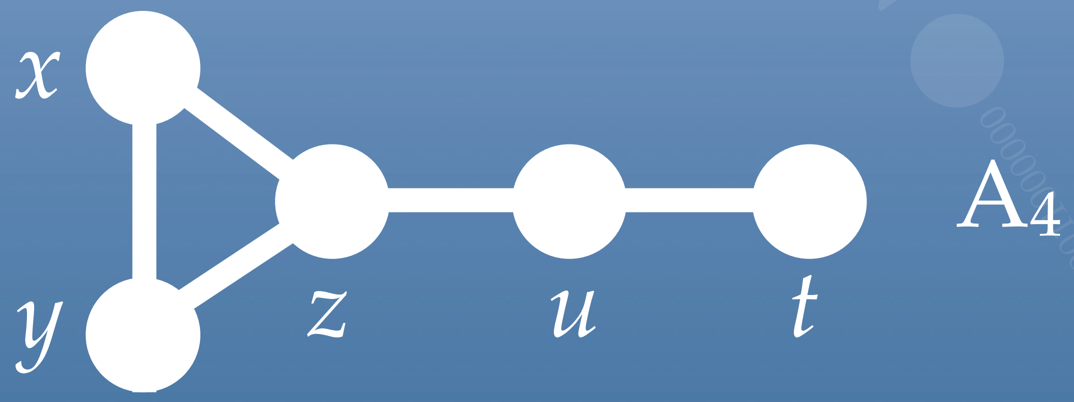
Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts: x, y, z, u, t.



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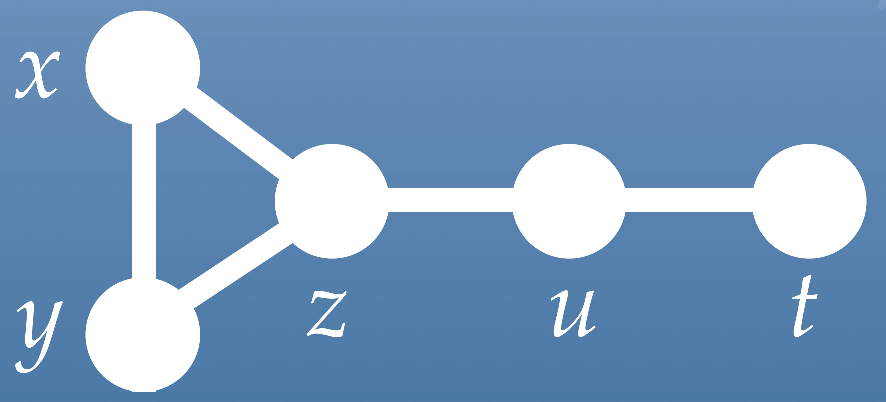
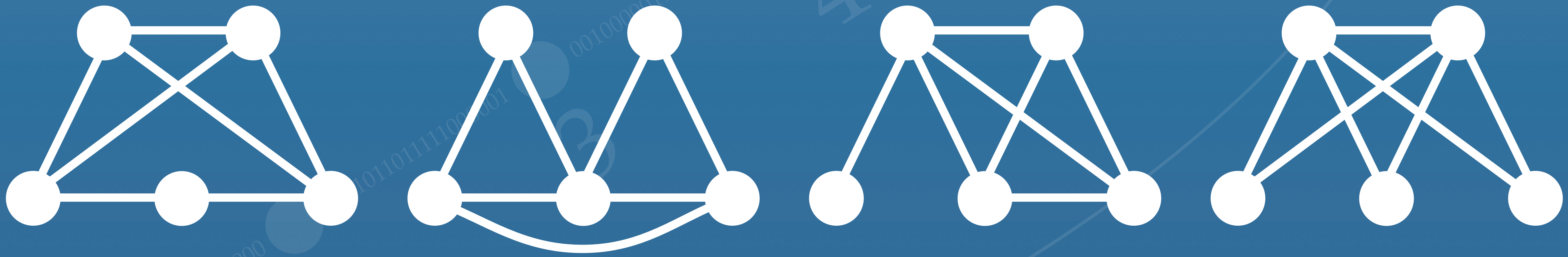


A_4

Leave out three edges:



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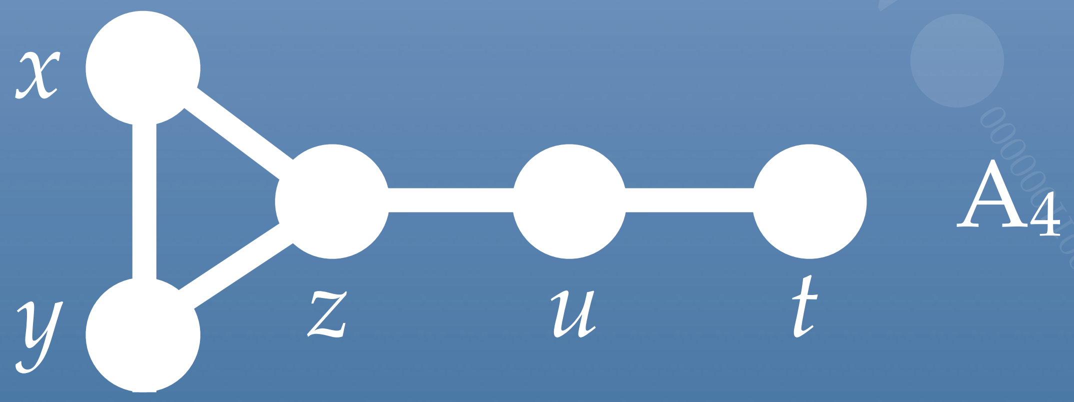

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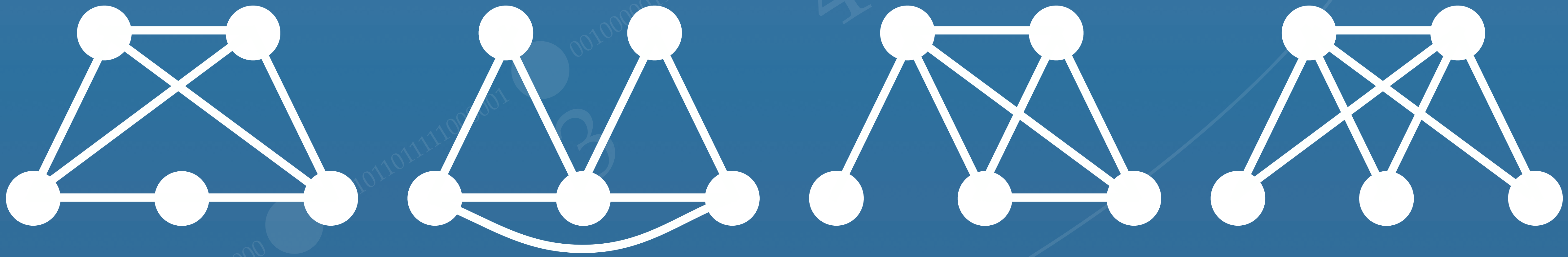

78-dim

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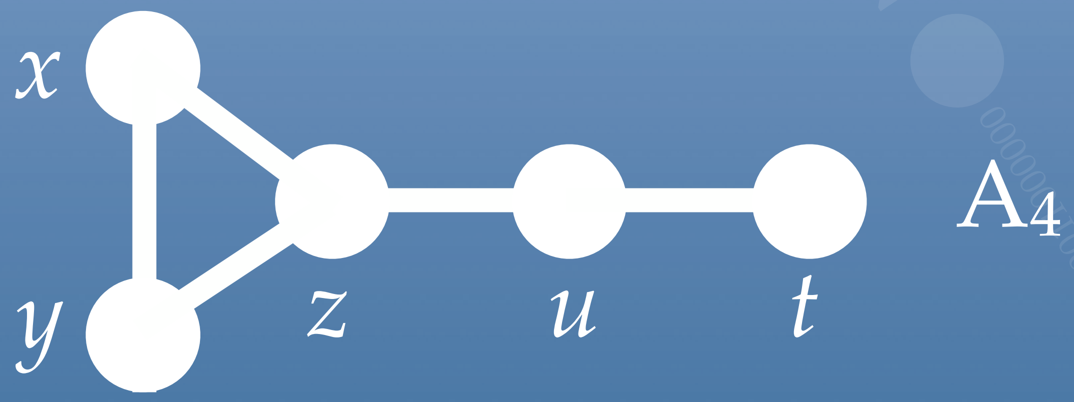

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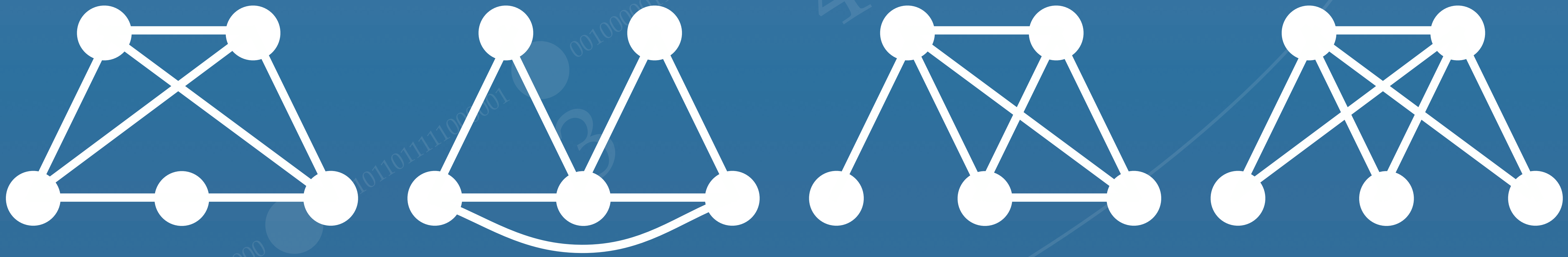

78-dim E_6

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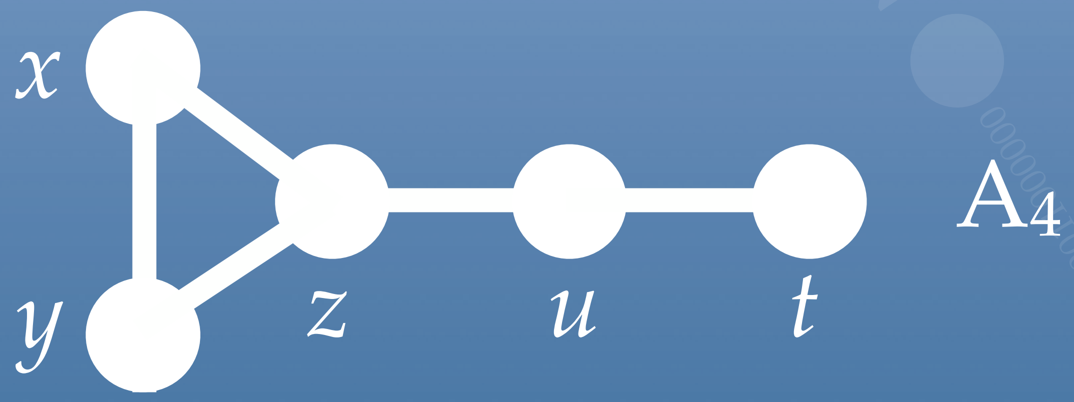

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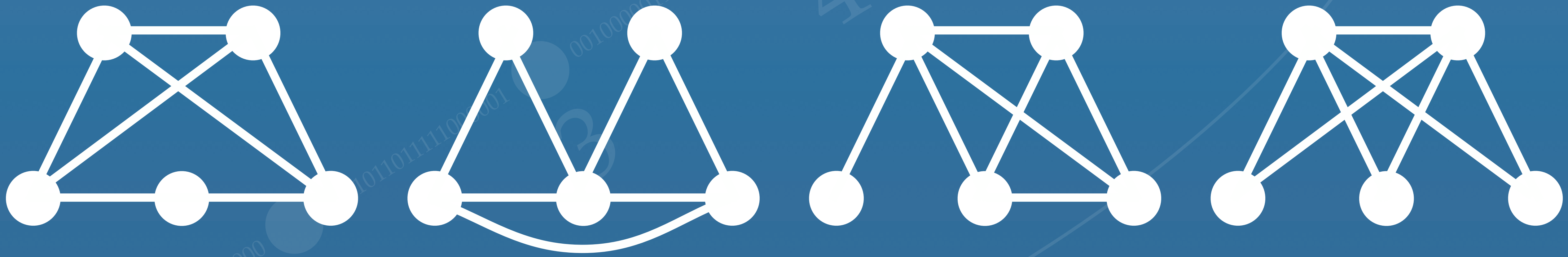

78-dim E_6 78-dim E_6

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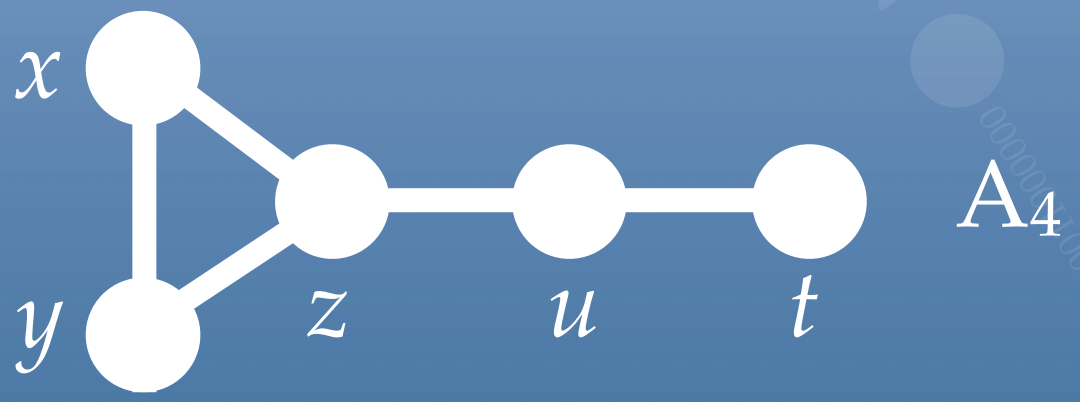
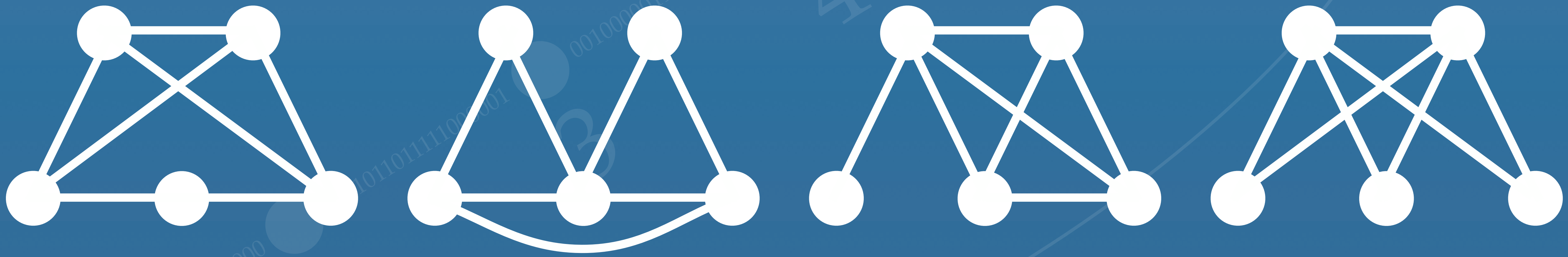


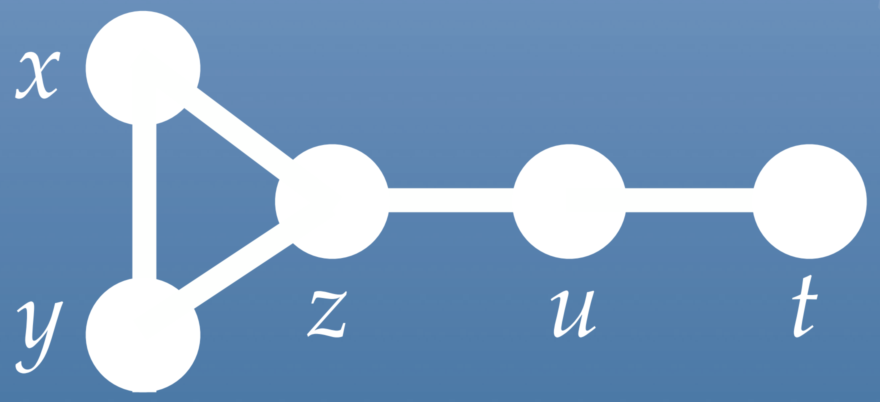
Leave out three edges:



78-dim E_6 78-dim E_6 78-dim E_6

Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts: x, y, z, u, t.



A_4

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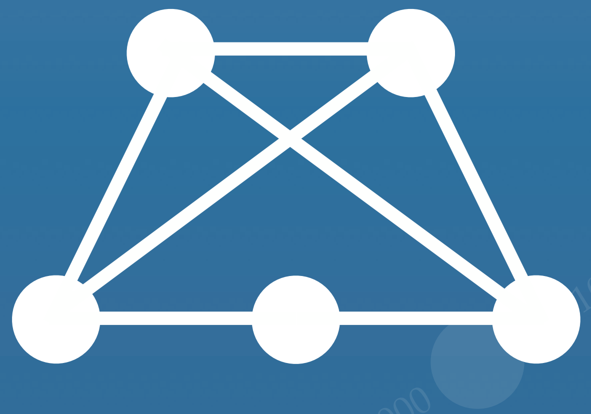

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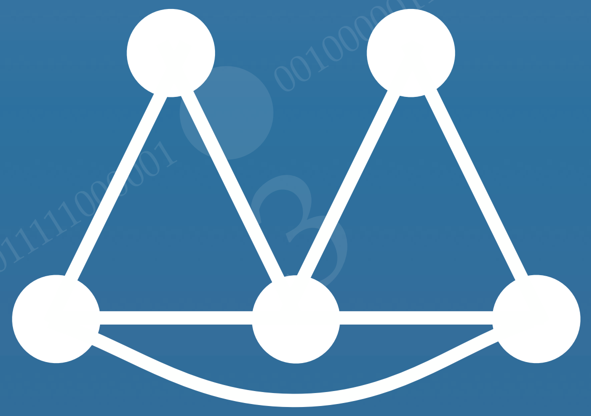

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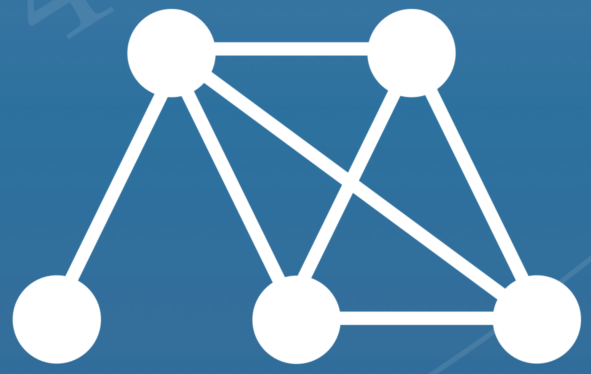

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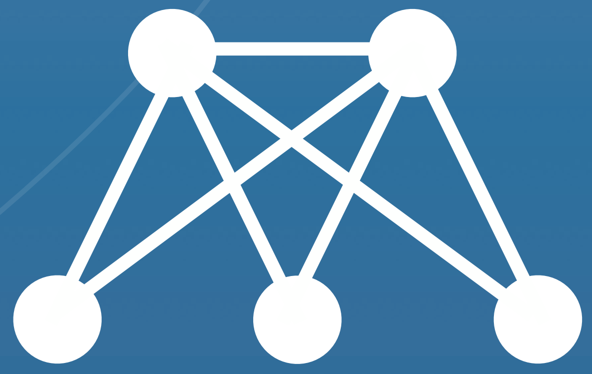
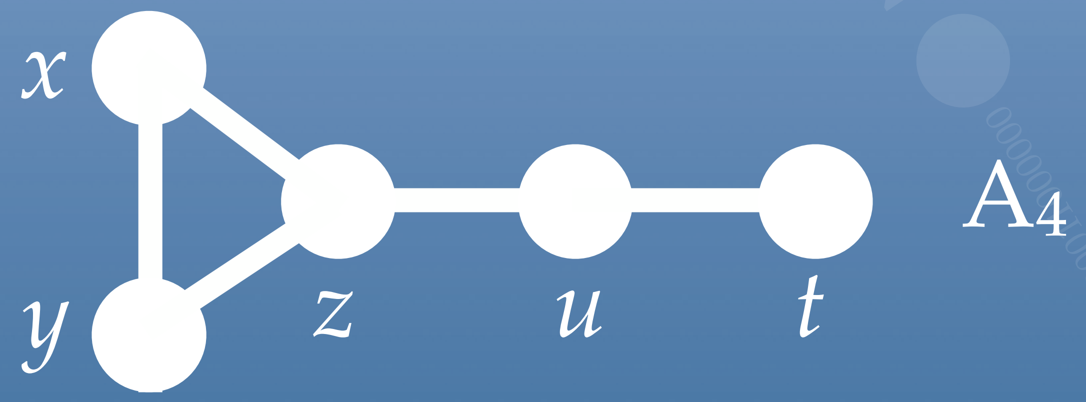


86-dim

Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts: x, y, z, u, t.



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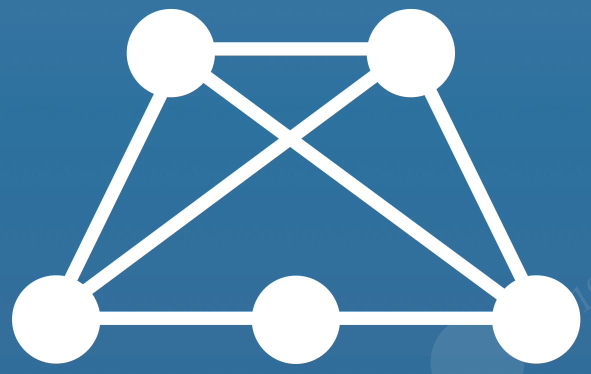
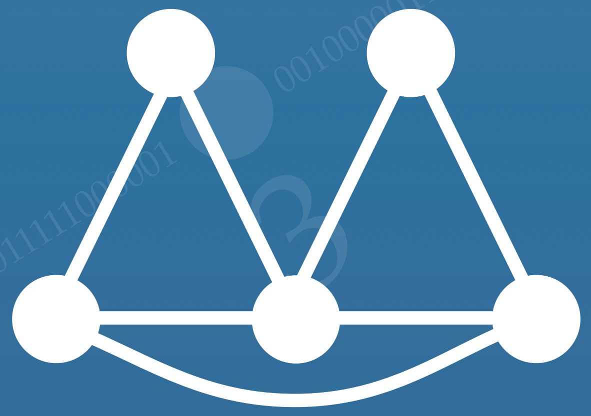
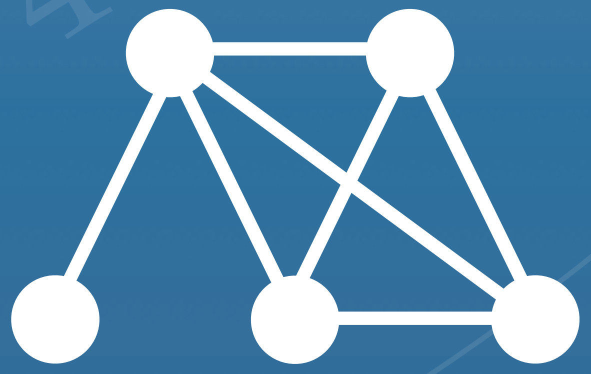


78-dim E_6



78-dim E_6



78-dim E_6

86-dim D_4 / ..
B_3 / ..

Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts

Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts

n	dim	type
2	3	A_1
3	8	A_2
4	28	D_4

Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts

537

n	dim	type
2	3	A_1
3	8	A_2
4	28	D_4
5	537	???

Finally: 5 generators

Lie algebra L gen'd by 5 extr. elts

537

n	dim	type
2	3	A_1
3	8	A_2
4	28	D_4
5	537	???

Theorems (ZK1991 / CSUW2001)

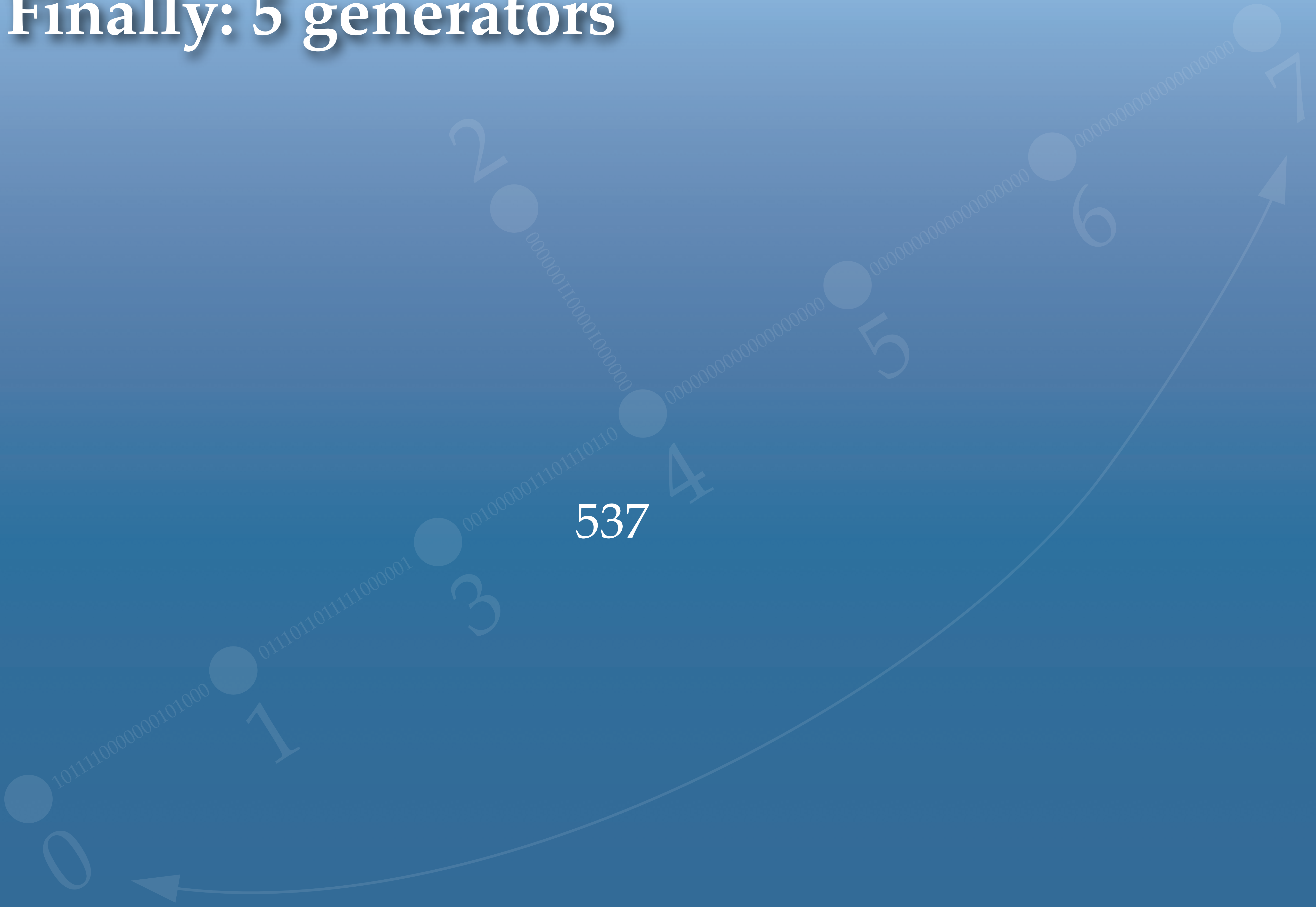
If L is a Lie algebra generated by extr. elts. then

- Minimal numbers of extremal generators:

type	num. gens	cond.
A_n	$n+1$	$n \geq 1$
B_n	$n+1$	$n \geq 3$
C_n	$2n$	$n \geq 2$
D_n	n	$n \geq 4$

type	num. gens	cond.
E_n	5	$n = 6, 7, 8$
F_4	5	
G_2	4	

Finally: 5 generators



537

Finally: 5 generators

? ?

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537

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