Construction of Chevalley Bases in all Characteristics

May 29, 2009

Euler Institute for
E I D M A
Discrete Mathematics and its Applications

DIAMANT
Discrete Interactive and Algorithmic Mathematics Algebra and Number Theory
A mathematics cluster funded by NWO OCW and EZ

Joint work with
On the occasion of Arjeh Cohen 's 60th birthday

Dan Roozemond

/ department of mathematics and computer science

TU/e Technische Universiteit Eindhoven University of Technology

Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- What next?

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What is a Lie Algebra?

- **Vector space:** $\mathbb{F}^n$

What is a Lie Algebra?

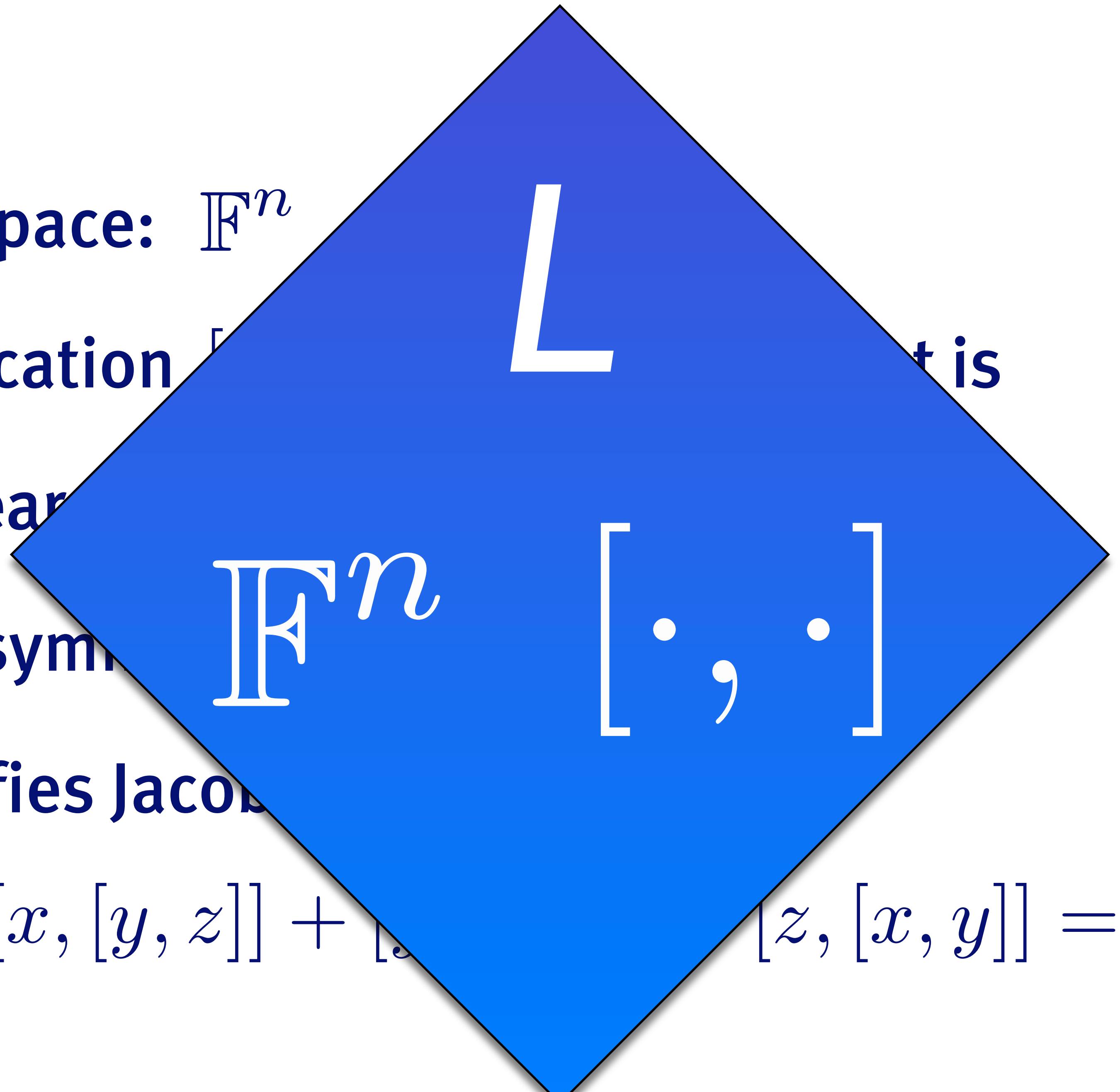
- Vector space: $\mathbb{F}^n$
- Multiplication $[\cdot,\cdot] : L \times L \mapsto L$ that is
  - Bilinear,
  - Anti-symmetric,
  - Satisfies Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

What is a Lie Algebra?

- Vector space: $\mathbb{F}^n$
- Multiplication [... t is
  - Bilinear
  - Anti-sym
  - Satisfies Jacob

$$[x,[y,z]] + [\quad [z,[x,y]] = 0$$



Simple Lie algebras

Classification (Killing, Cartan)

If $\mathrm{char}(\mathbb{F}) = 0$ and $\mathbb{F}$ algebraically closed, then the only simple Lie algebras are:

$$\mathrm{A}_n \ (n \geq 1) \qquad \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$$

$$\mathrm{B}_n \ (n \geq 2) \qquad \mathrm{F}_4$$

$$\mathrm{C}_n \ (n \geq 3) \qquad \mathrm{G}_2$$

$$\mathrm{D}_n \ (n \geq 4)$$

Why Study Lie Algebras?

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- Study *groups* by their Lie algebras:
  - Simple algebraic group G <-> Unique Lie algebra L
  - Many properties carry over to L
  - Easier to calculate in L
  - $G \leq \mathrm{Aut}(L)$, often even $G = \mathrm{Aut}(L)$

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- Opportunities for:
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  - ...
- Because there are problems to be solved!
  - ... and a thesis to be written...

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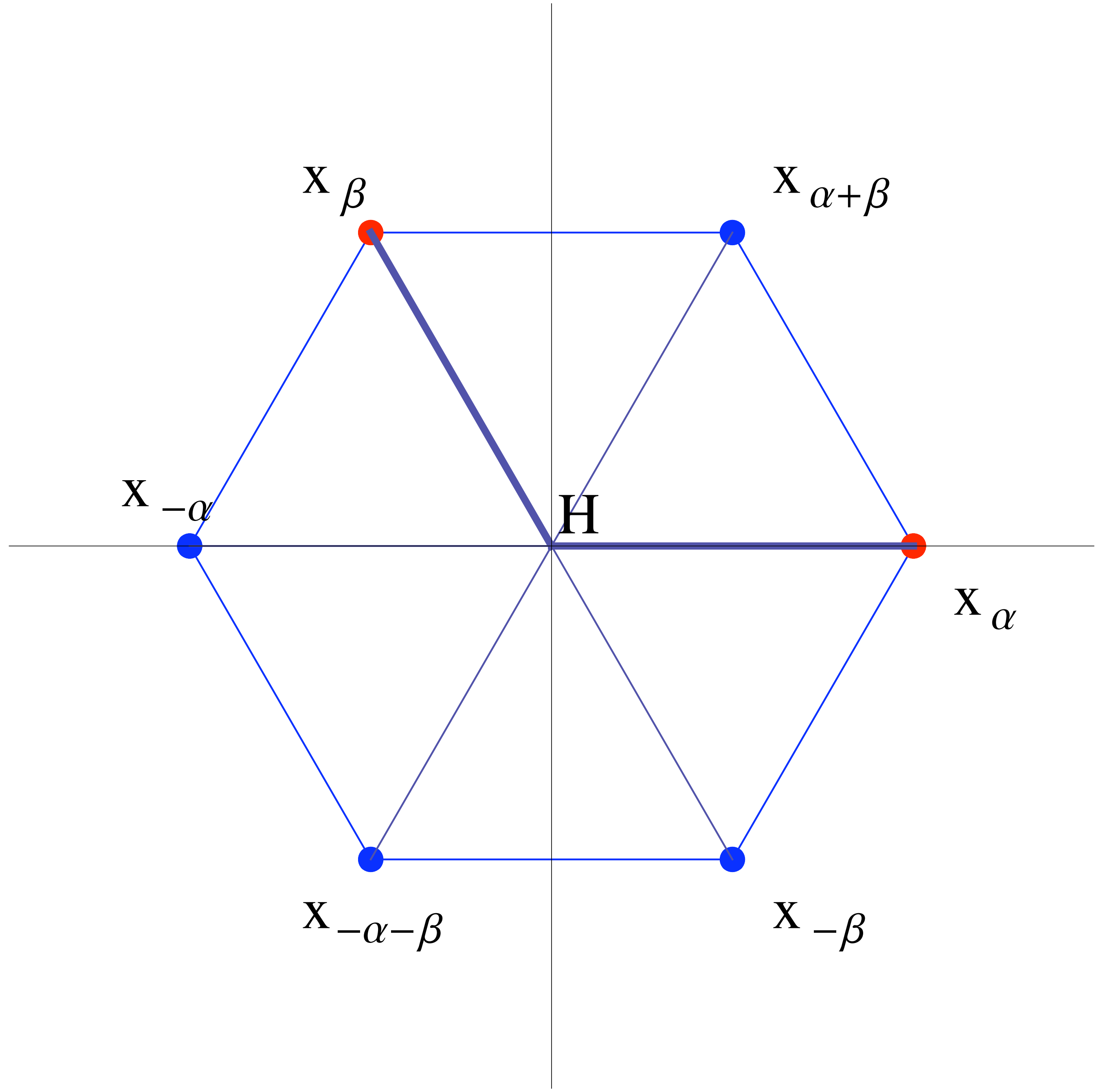
Chevalley Bases



Many Lie algebras have a *Chevalley basis*!

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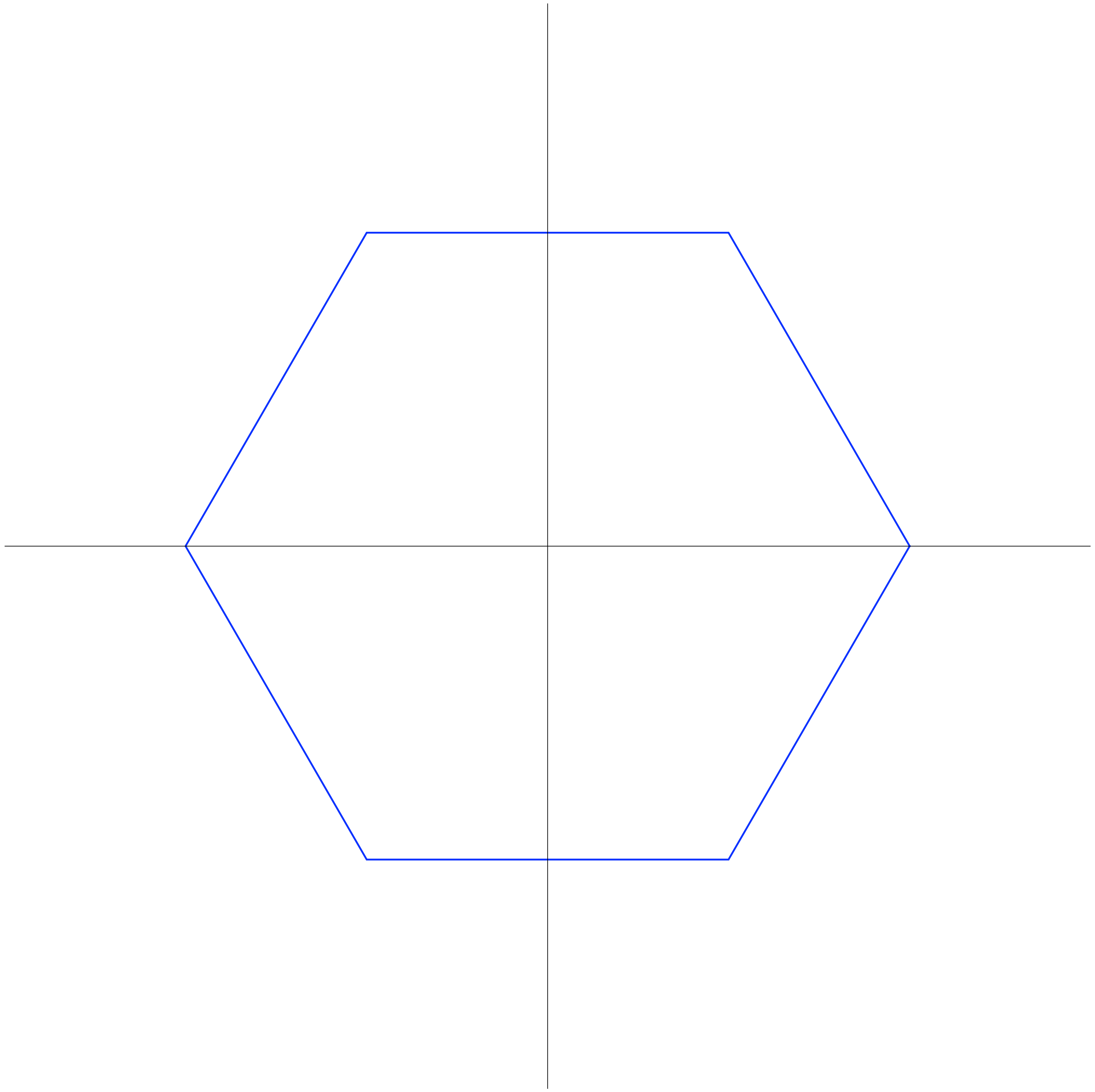
- **A hexagon**



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Root Systems

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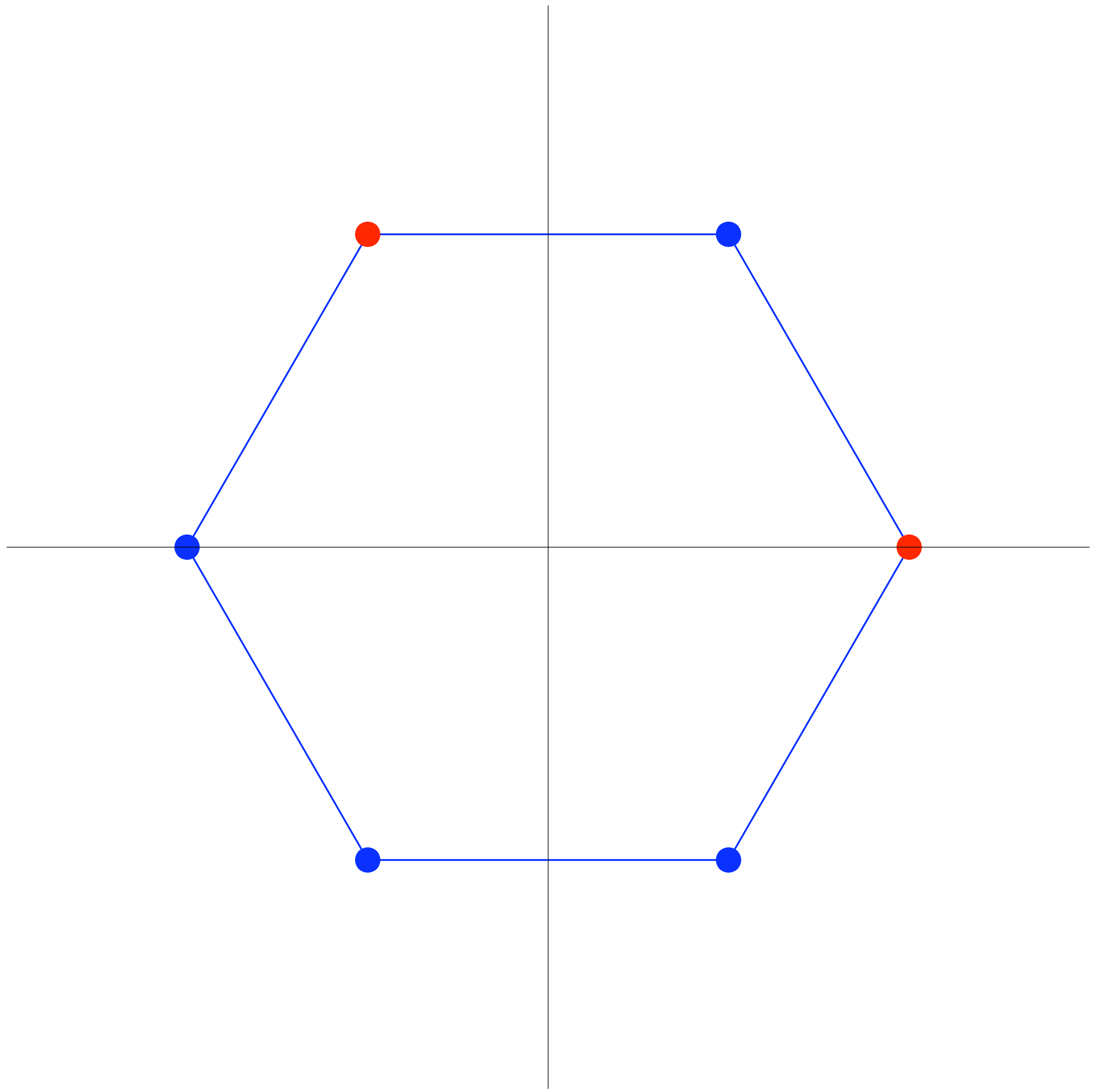

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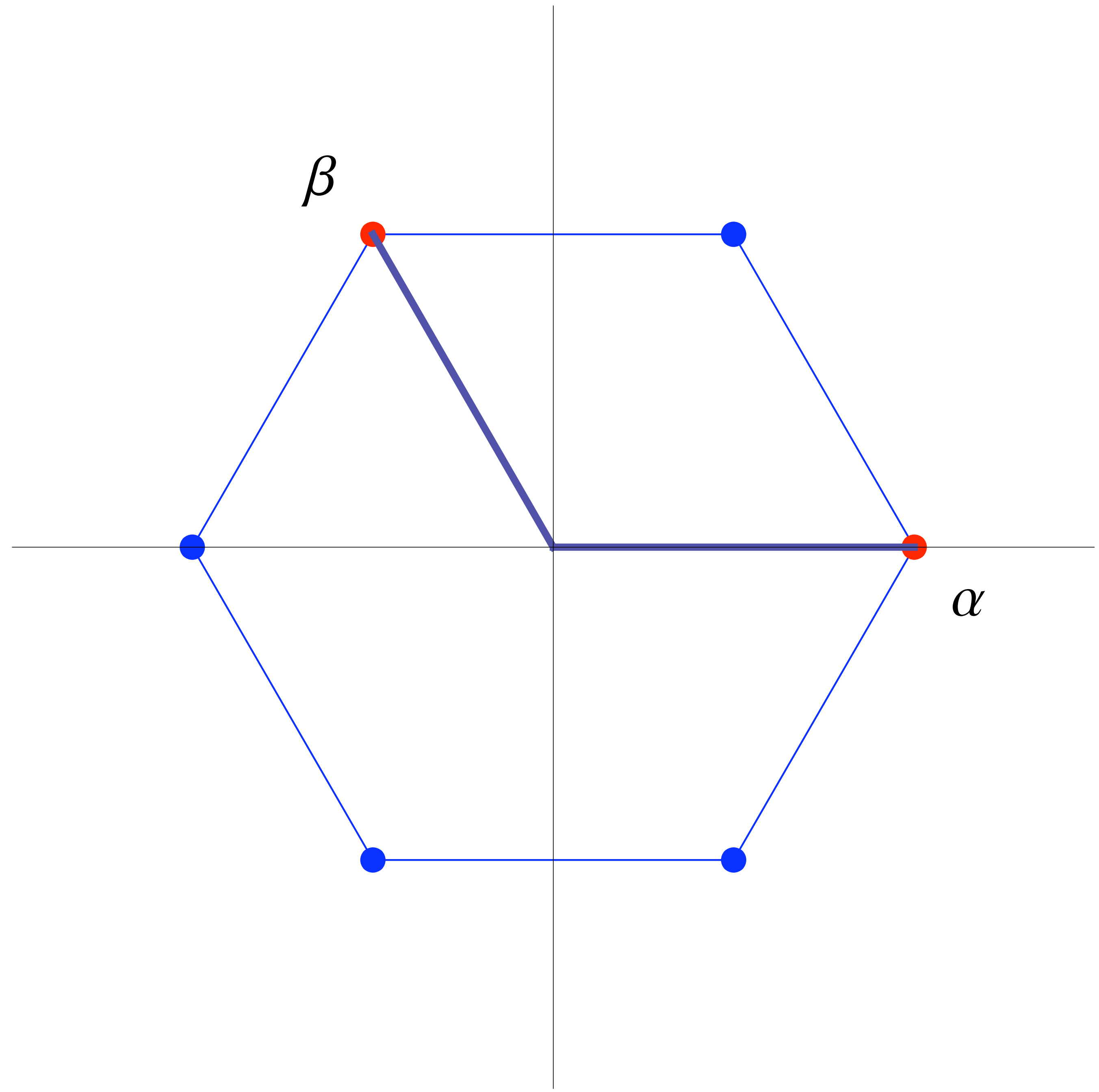
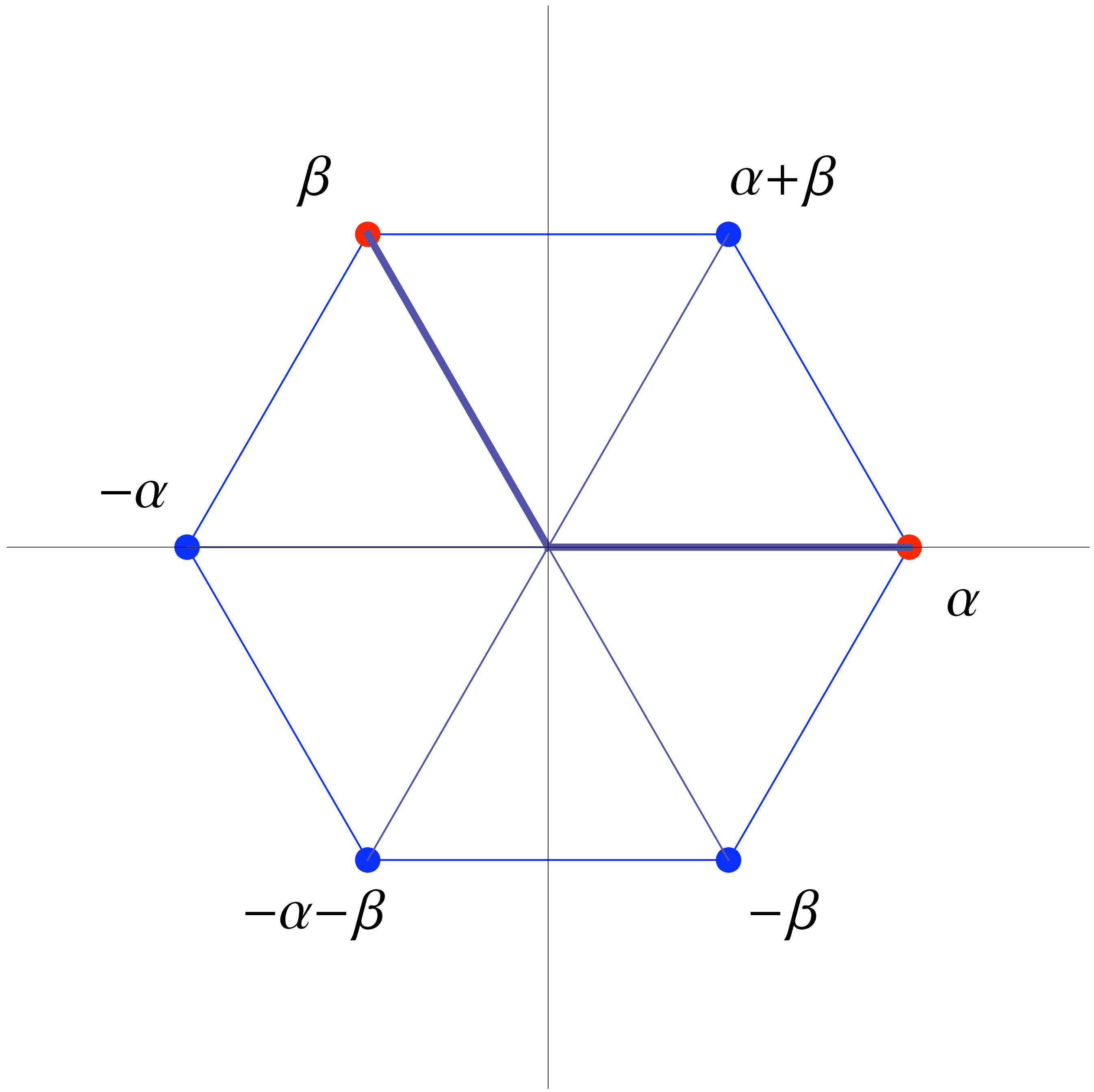


Root Systems

- A hexagon
- A root system of type A_2



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Root Data

Definition (Root Datum)

$$R = (X, \Phi, Y, \Phi^\vee), \quad \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$$

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- put in duality by $\langle \cdot, \cdot \rangle$,
- $\Phi \subseteq X$: roots,
- $\Phi^\vee \subseteq Y$: coroots.

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One **Root System** $\longrightarrow$ $\left\{ \begin{array}{c} \textit{Several} \textbf{ Root Data:} \\ \text{"adjoint"} \\ \vdots \\ \text{"simply connected"} \end{array} \right.$

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One **Root System** $\longrightarrow$ *Several* **Root Data:**

"adjoint"

⋮

"simply connected"

Irreducible Root Data: $\mathrm{A}_n^{\cdot}, \mathrm{B}_n^{\cdot}, \mathrm{C}_n^{\cdot}, \mathrm{D}_n^{\cdot}, \mathrm{E}_6^{\cdot}, \mathrm{E}_7^{\cdot}, \mathrm{E}_8^{\cdot}, \mathrm{F}_4^{\cdot}, \mathrm{G}_2^{\cdot}.$

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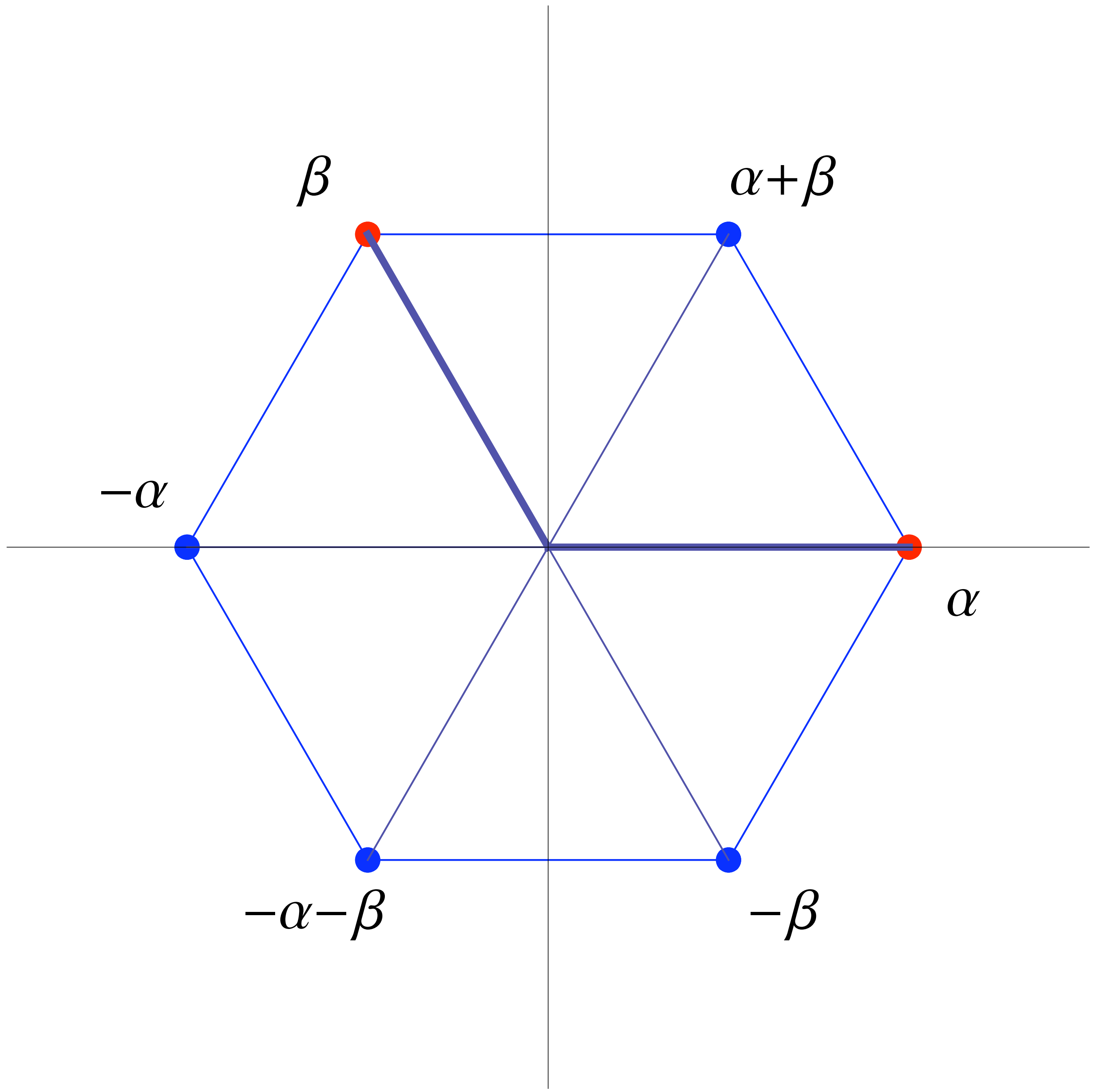
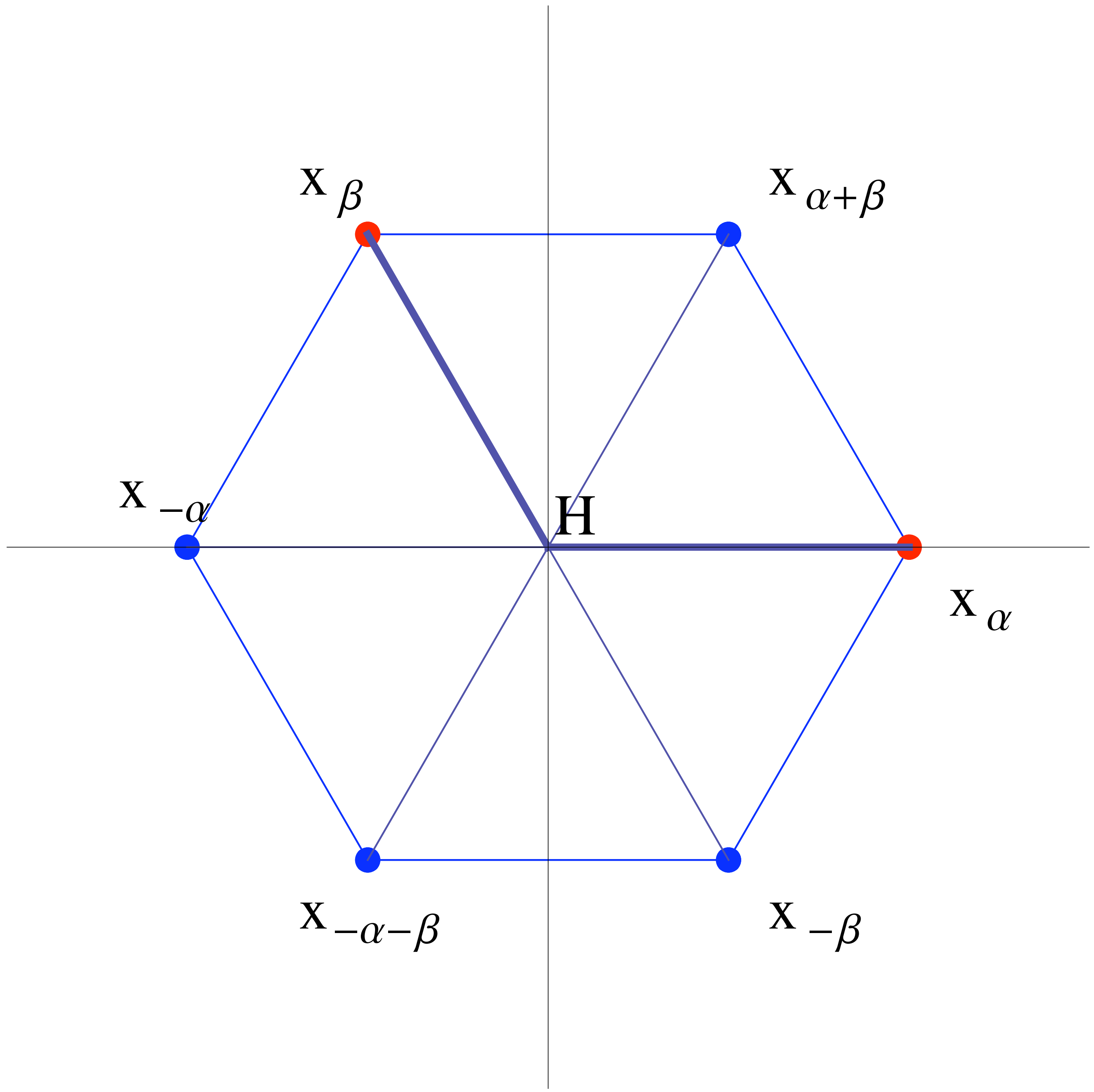


Root Data

- A hexagon
- A root system of type A_2
- A Lie algebra of type A_2



Chevalley Basis

Definition (Chevalley Basis)

Formal basis: $L = \bigoplus_{i=1,\ldots,n} \mathbb{F}h_i \oplus \bigoplus_{\alpha\in\Phi} \mathbb{F}x_\alpha$

Bilinear anti-symmetric multiplication satisfies ($i, j \in \{1, \ldots, n\}; \alpha, \beta \in \Phi$) :

$$\begin{aligned}
[h_i, h_j] &= 0, \\
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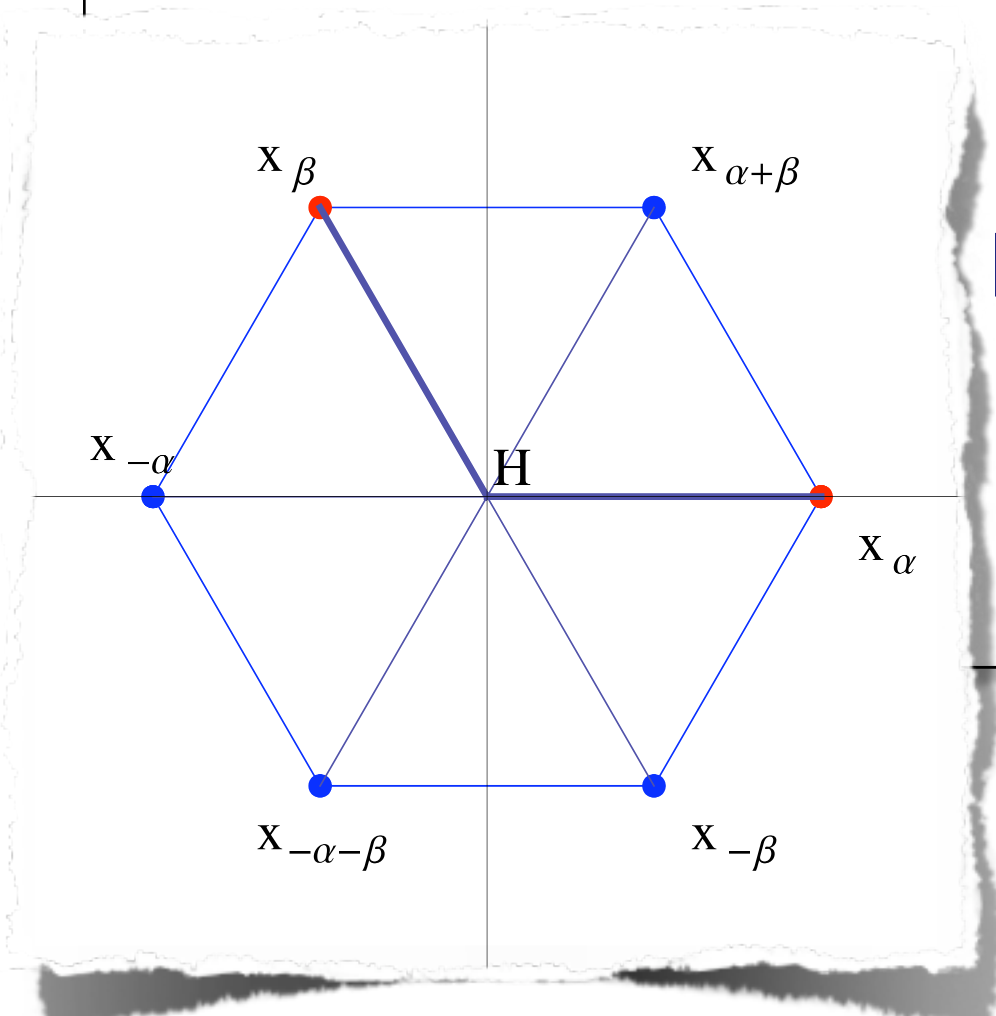

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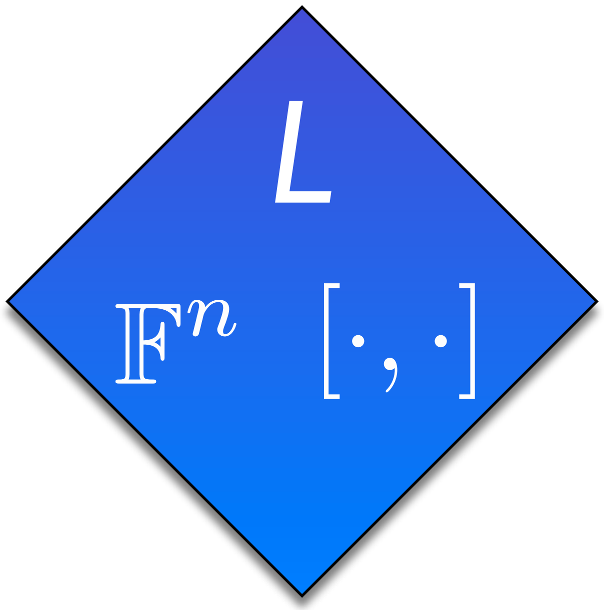
Why Chevalley bases?

- Because transformation between two Chevalley bases is an automorphism of L,
- So we can test isomorphism between two Lie algebras (and find isomorphisms!) by computing Chevalley bases.

Why?



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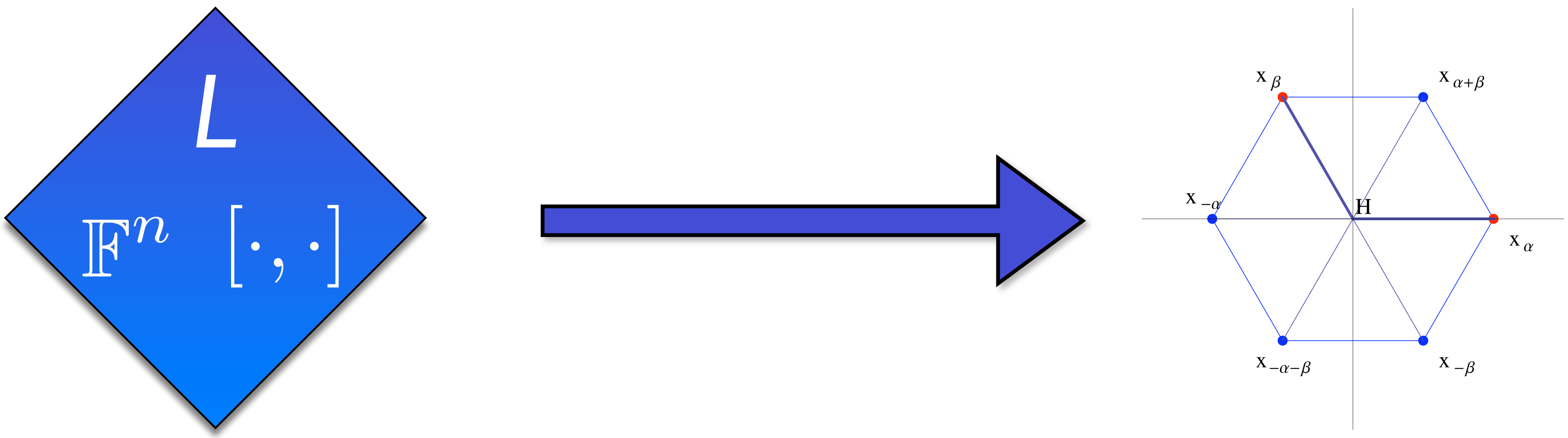

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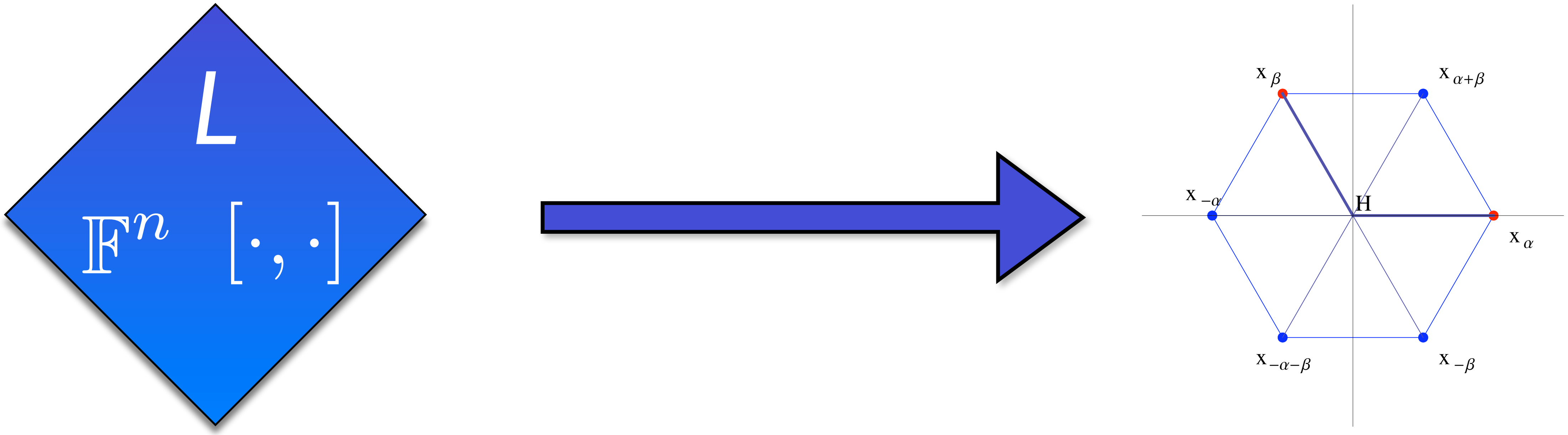
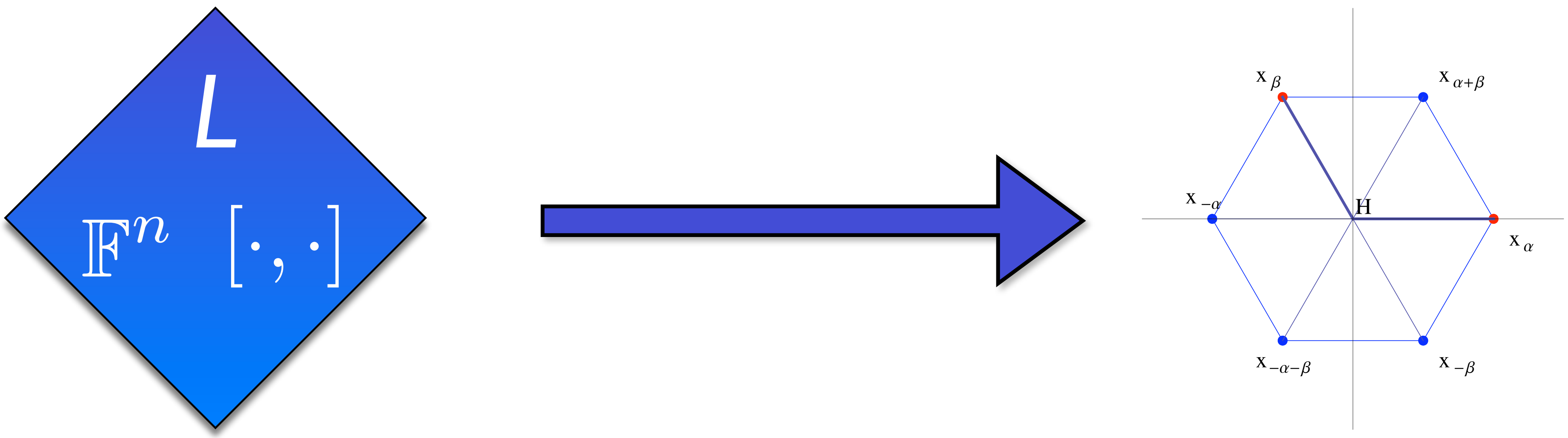

L
$\mathbb{F}^n$ $[\cdot,\cdot]$
x_β
$x_{\alpha+\beta}$
$x_{-\alpha}$
H
x_α
$x_{-\alpha-\beta}$
$x_{-\beta}$


$[\cdot,\cdot]$
L
$\mathbb{F}^n$

Why?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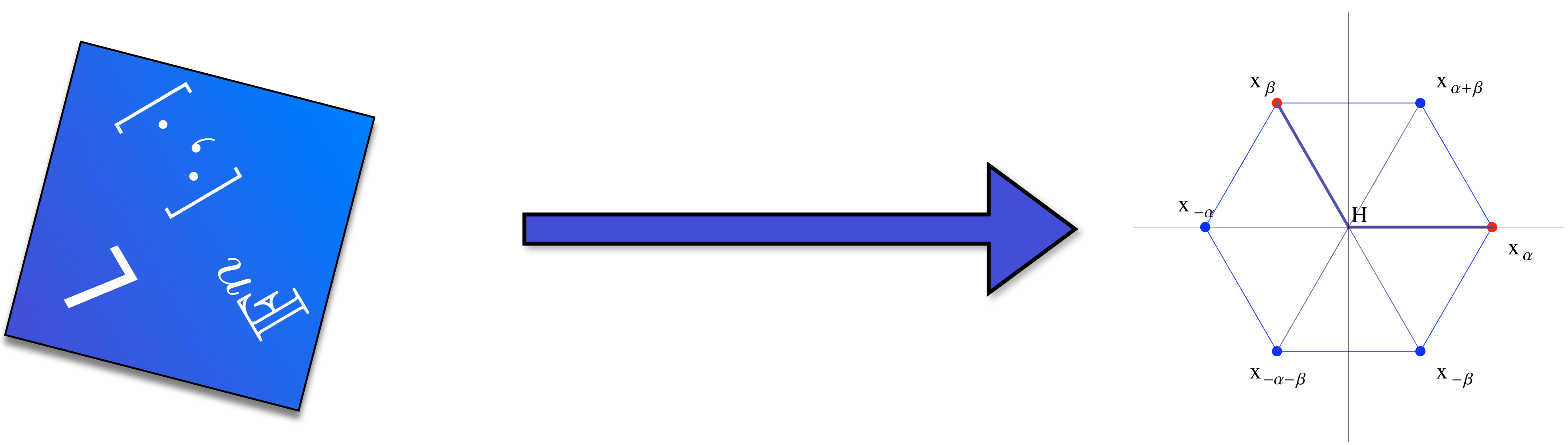
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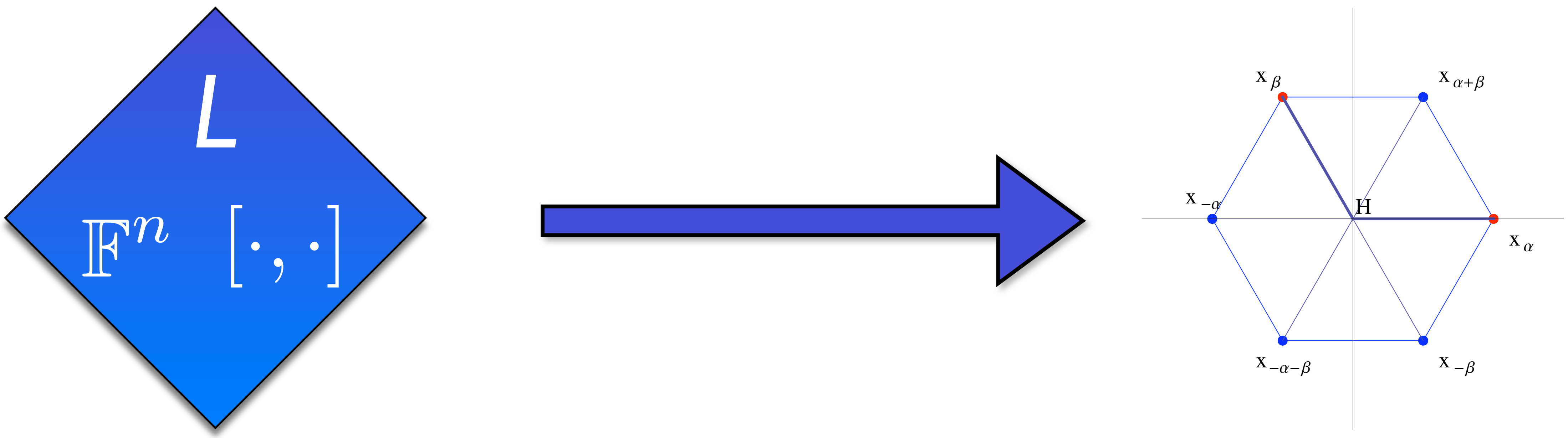
L
$\mathbb{F}^n$ $[\cdot,\cdot]$
x_β
$x_{\alpha+\beta}$
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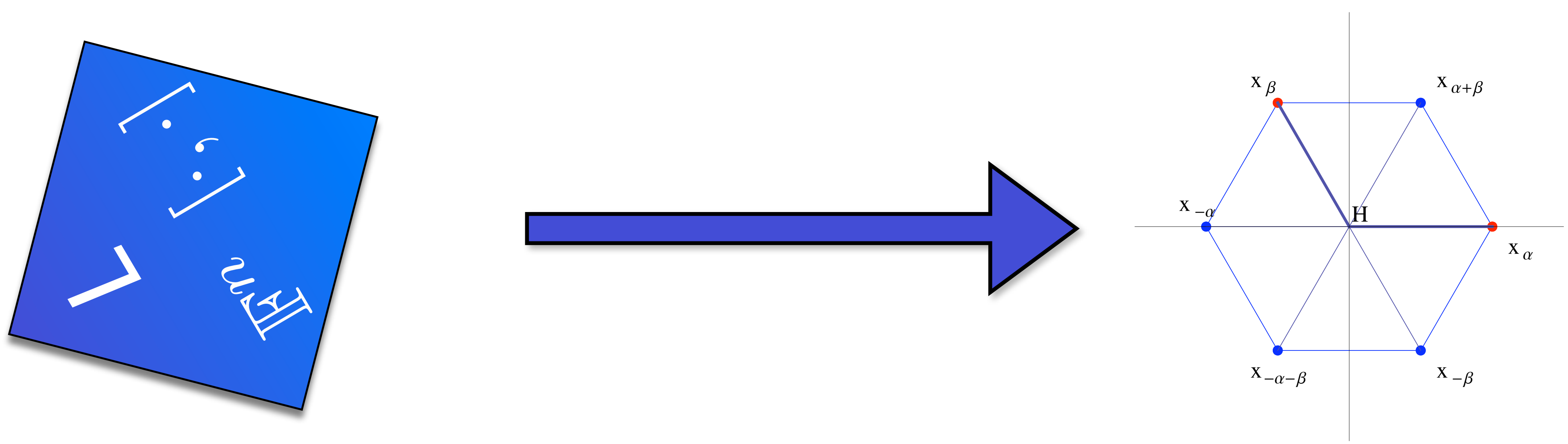
H
x_α
$x_{-\alpha-\beta}$
$x_{-\beta}$

equal


x_β
$x_{\alpha+\beta}$
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H
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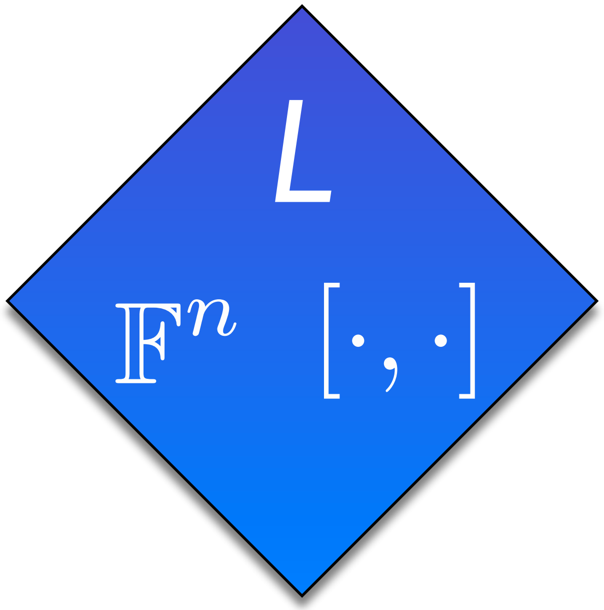
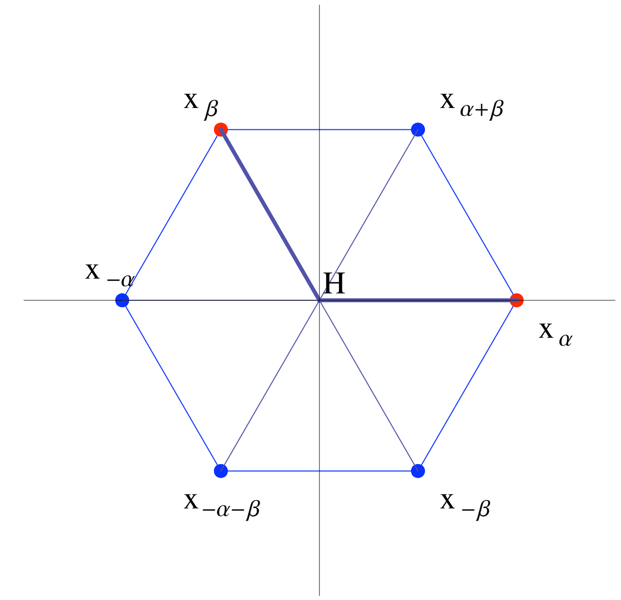
$x_{-\alpha-\beta}$
$x_{-\beta}$

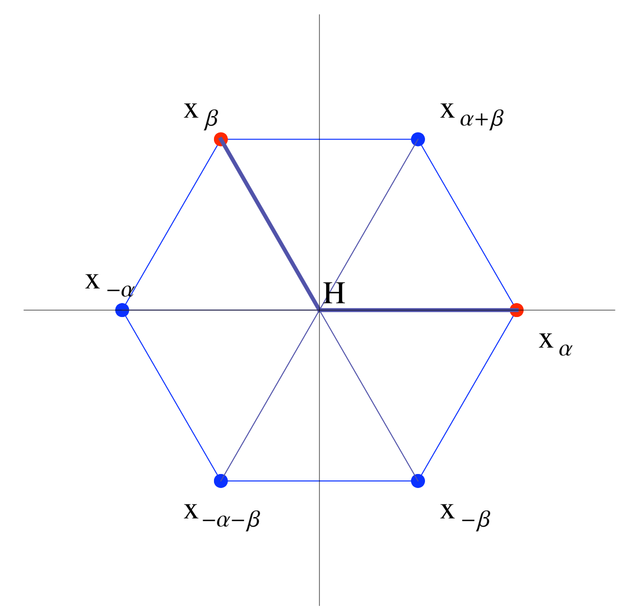
Why?





isomorphic!

equal



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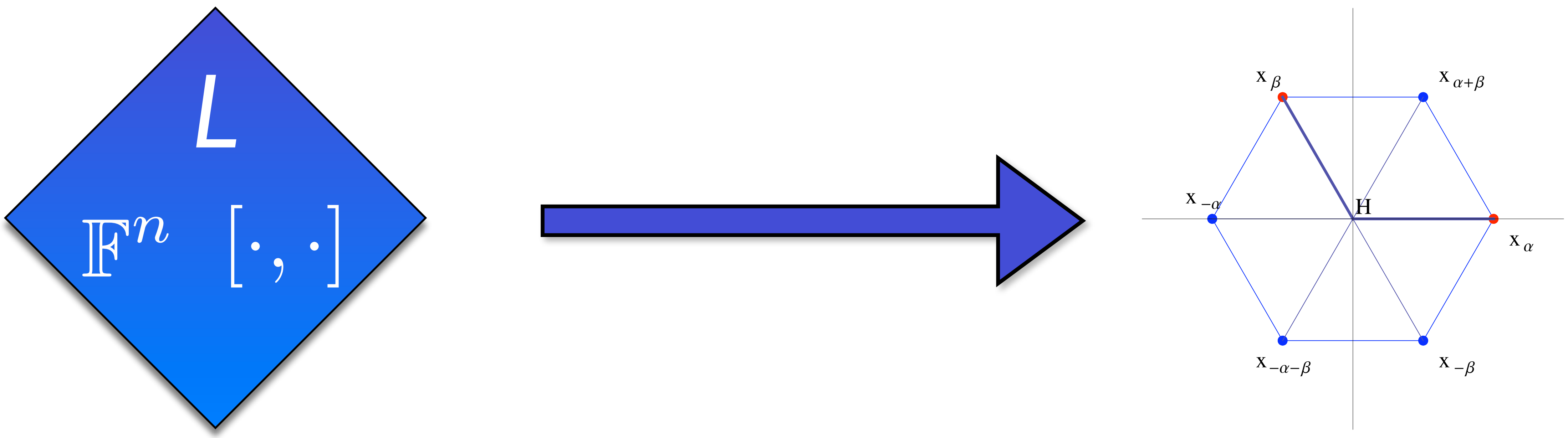
L
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x_β
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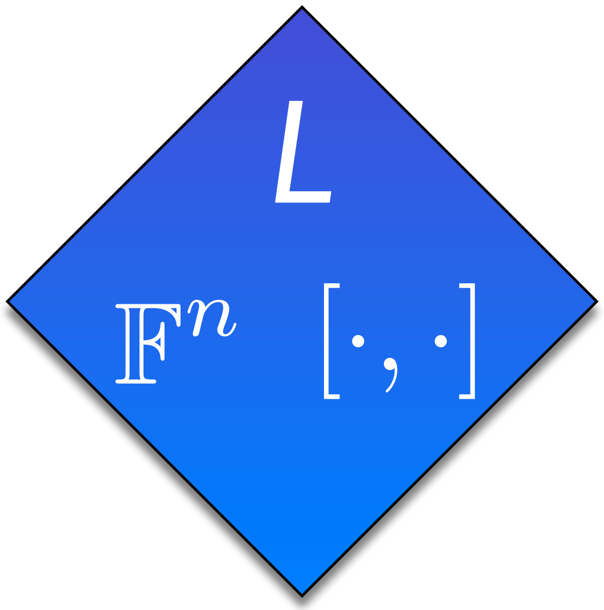
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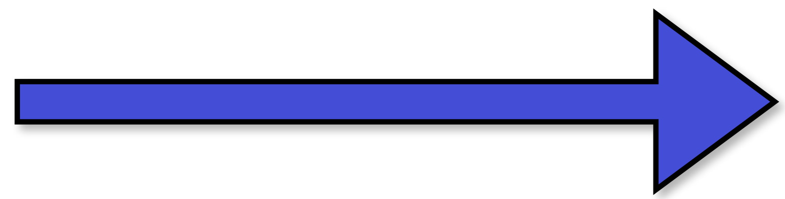

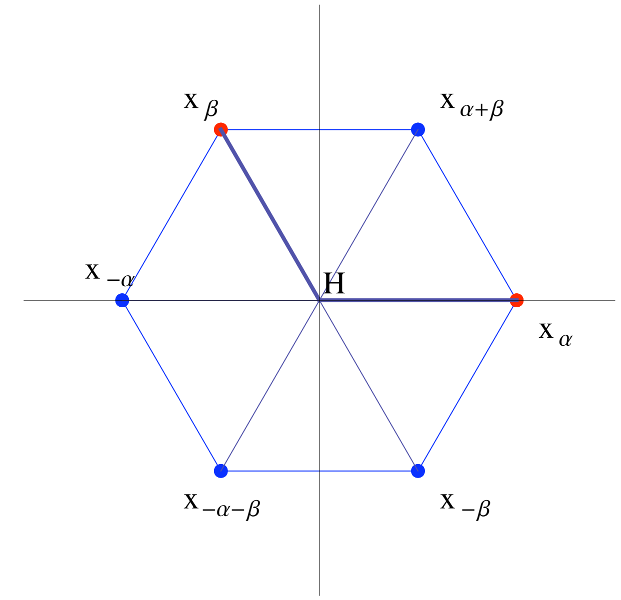
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H
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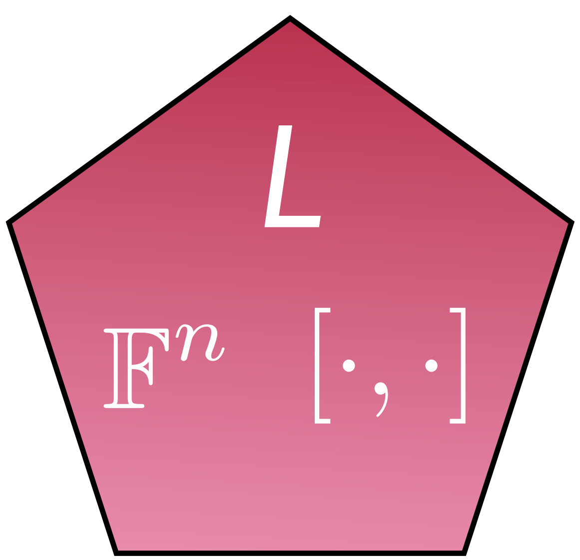
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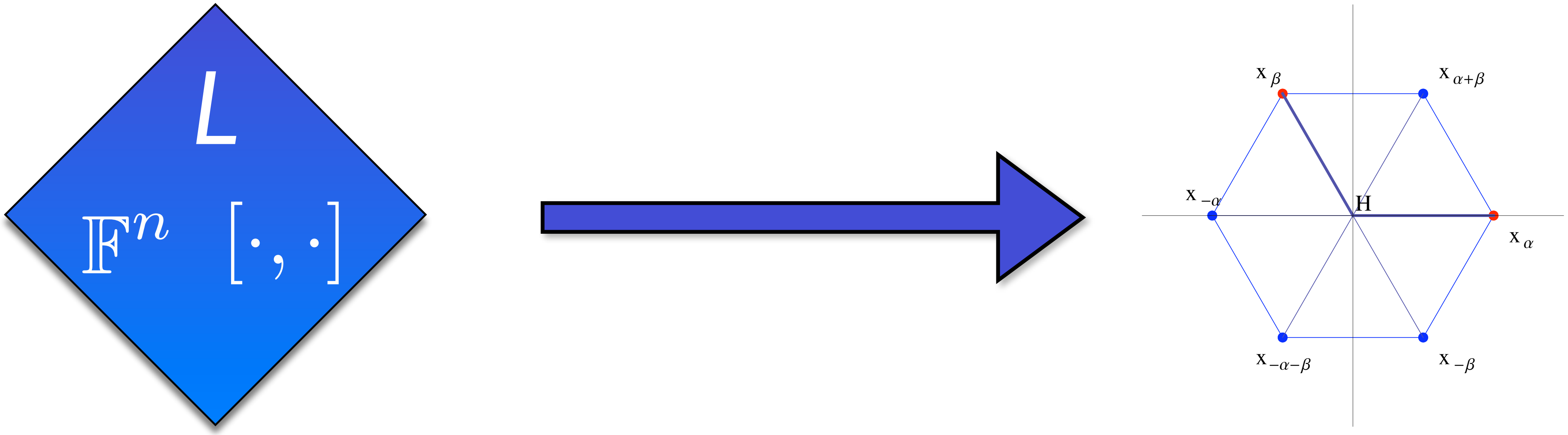
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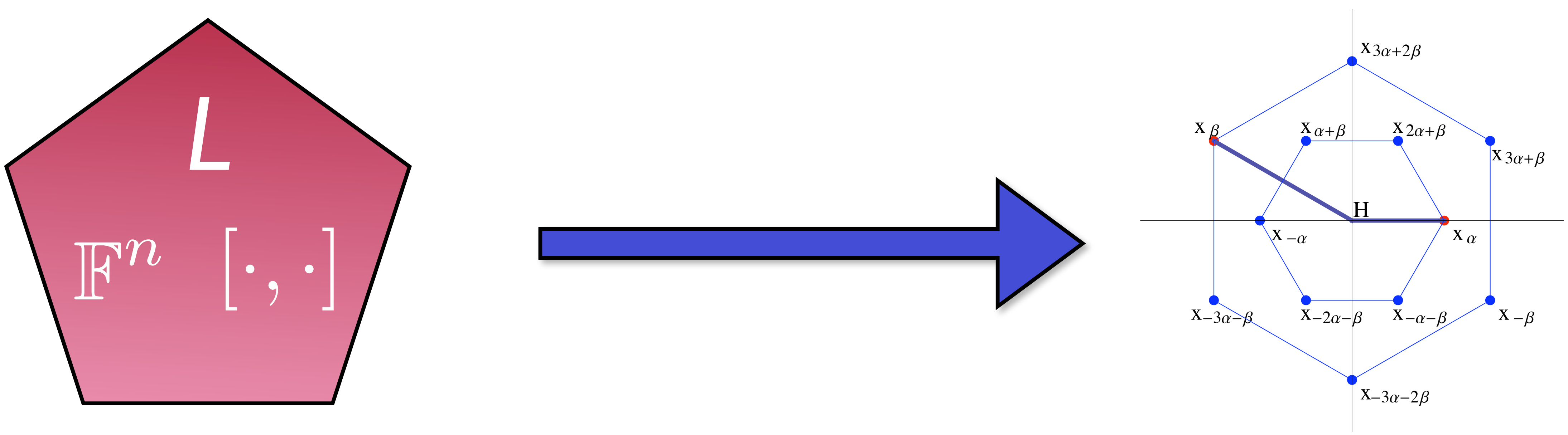
L
$\mathbb{F}^n$ $[\cdot,\cdot]$
$x_{3\alpha+2\beta}$
x_β
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$x_{2\alpha+\beta}$
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H
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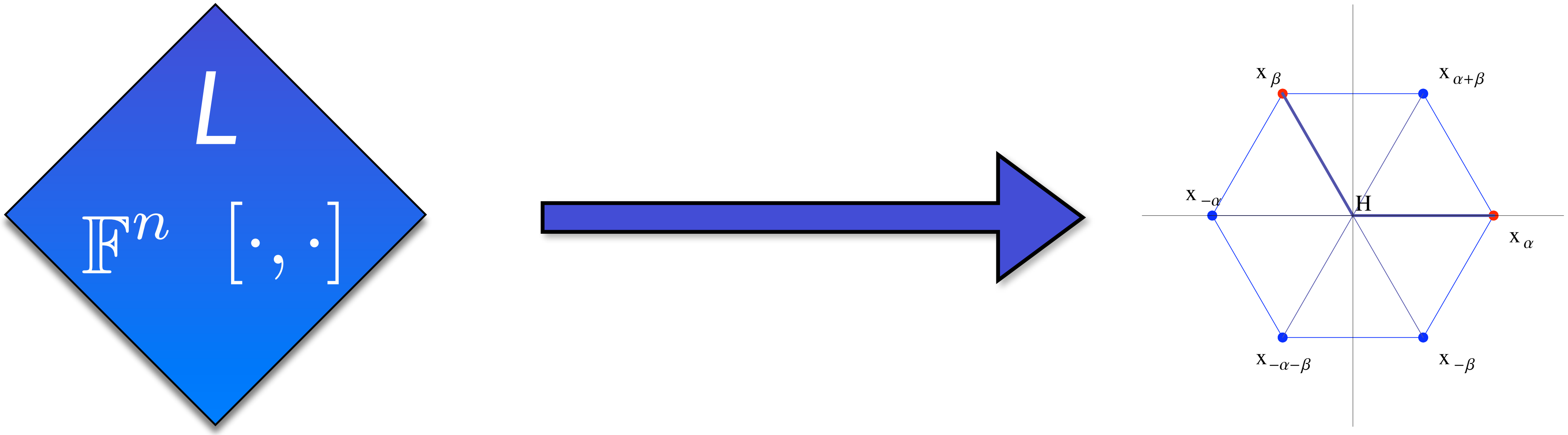
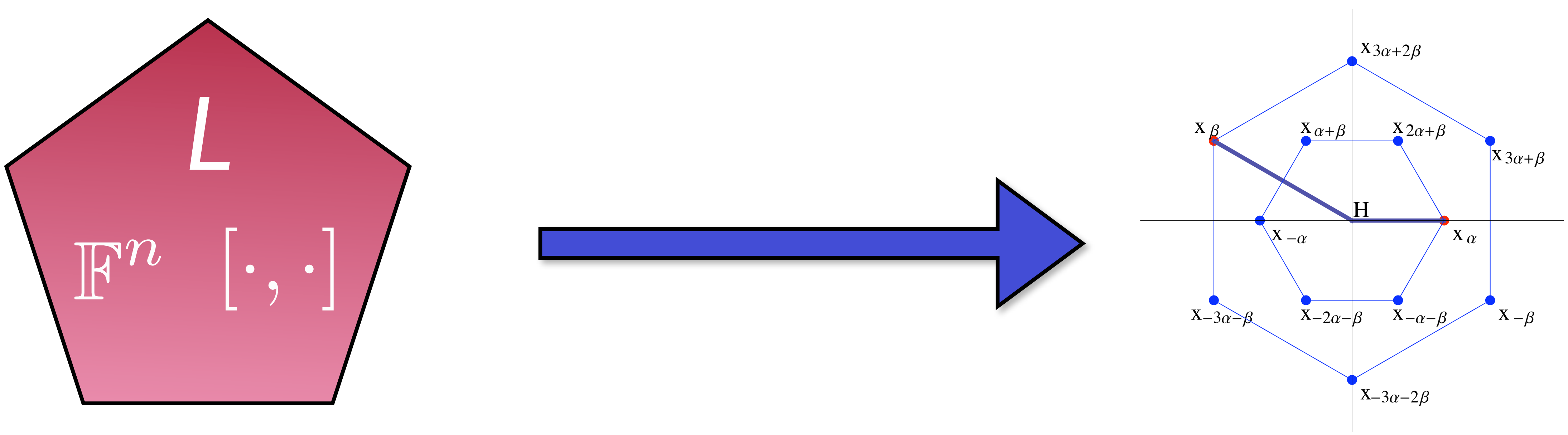
Why?



not equal



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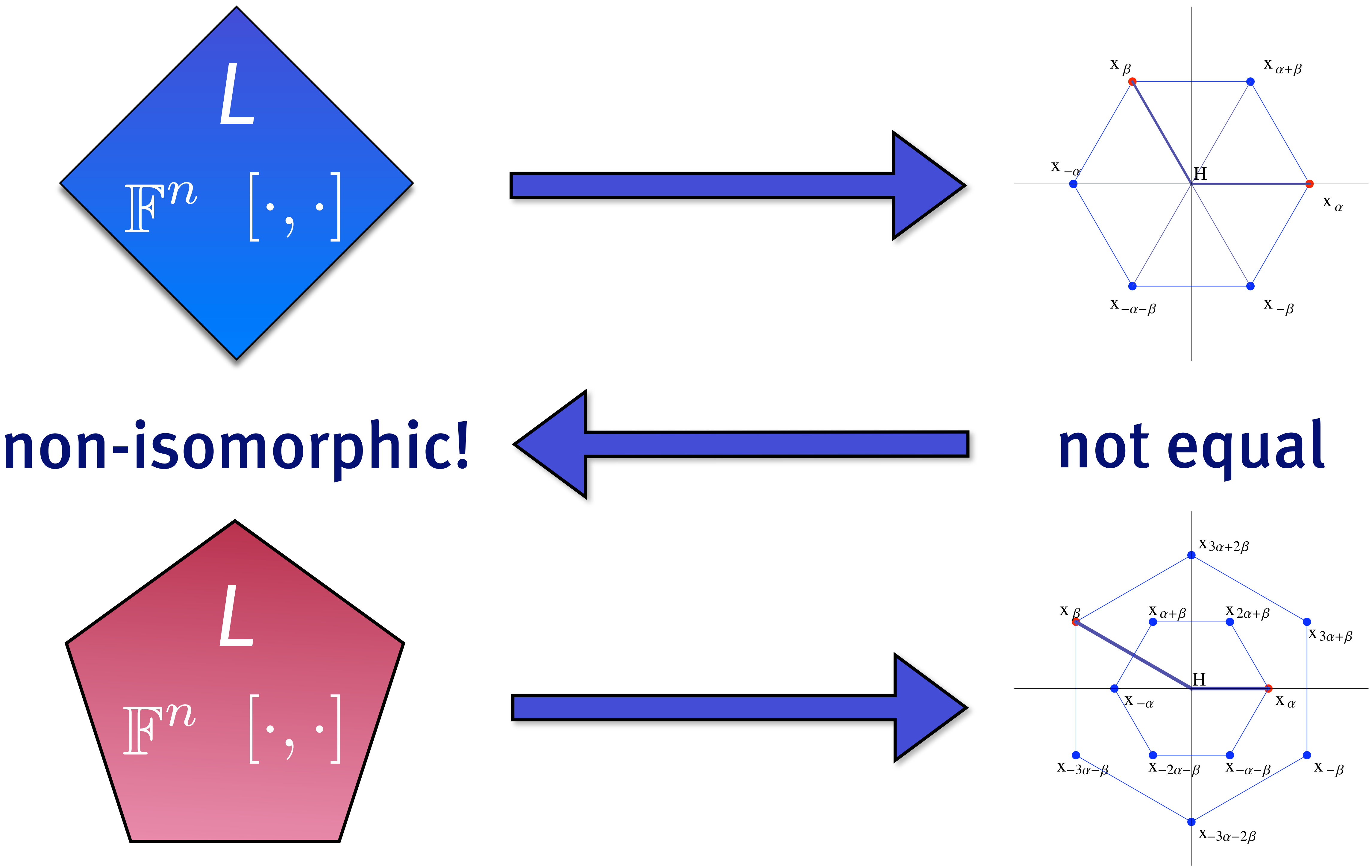
L
$\mathbb{F}^n$ $[\cdot,\cdot]$
x_β
$x_{\alpha+\beta}$
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H
x_α
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$x_{-\beta}$
non-isomorphic!
not equal
L
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- What is a Lie algebra?
- What is a Chevalley basis?
- **How to compute Chevalley bases?**
- **What next?**

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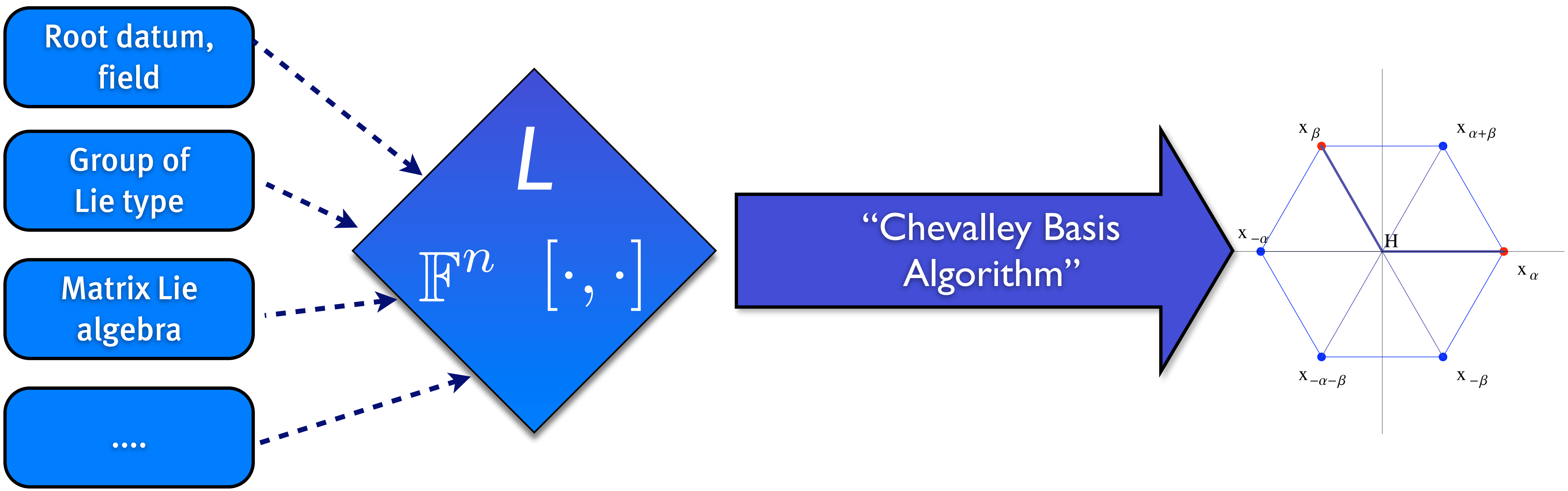
The Mission

- Given a Lie algebra (on a computer),
- Want to know which Lie algebra it is,
- So want to compute a Chevalley basis for it (if possible).



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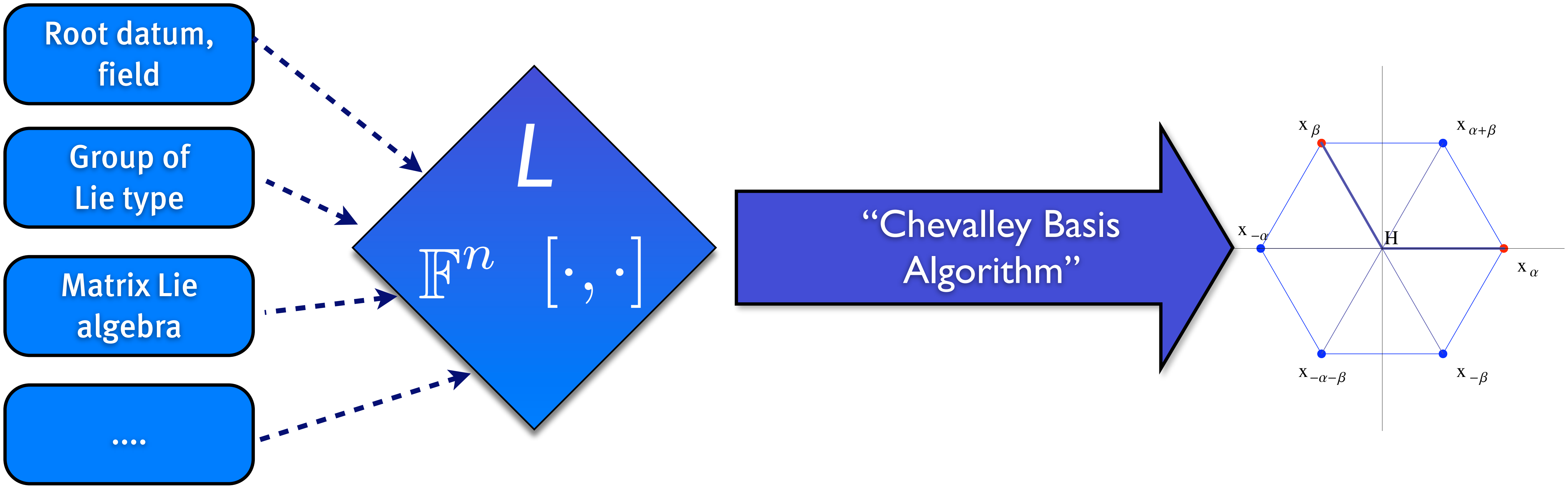
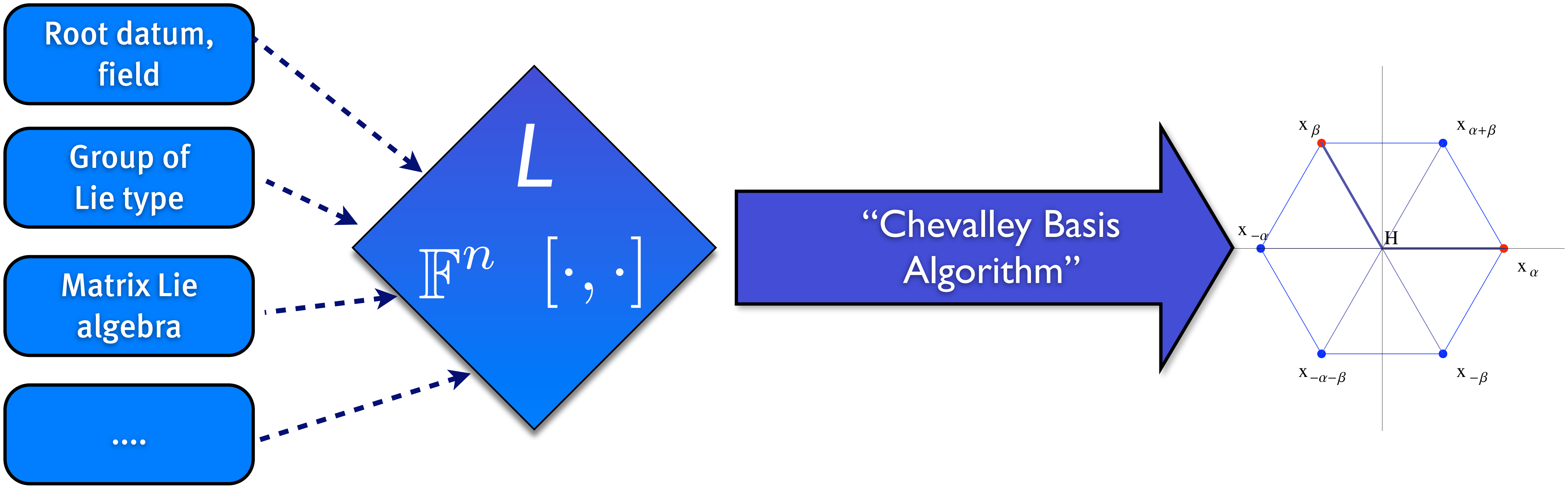


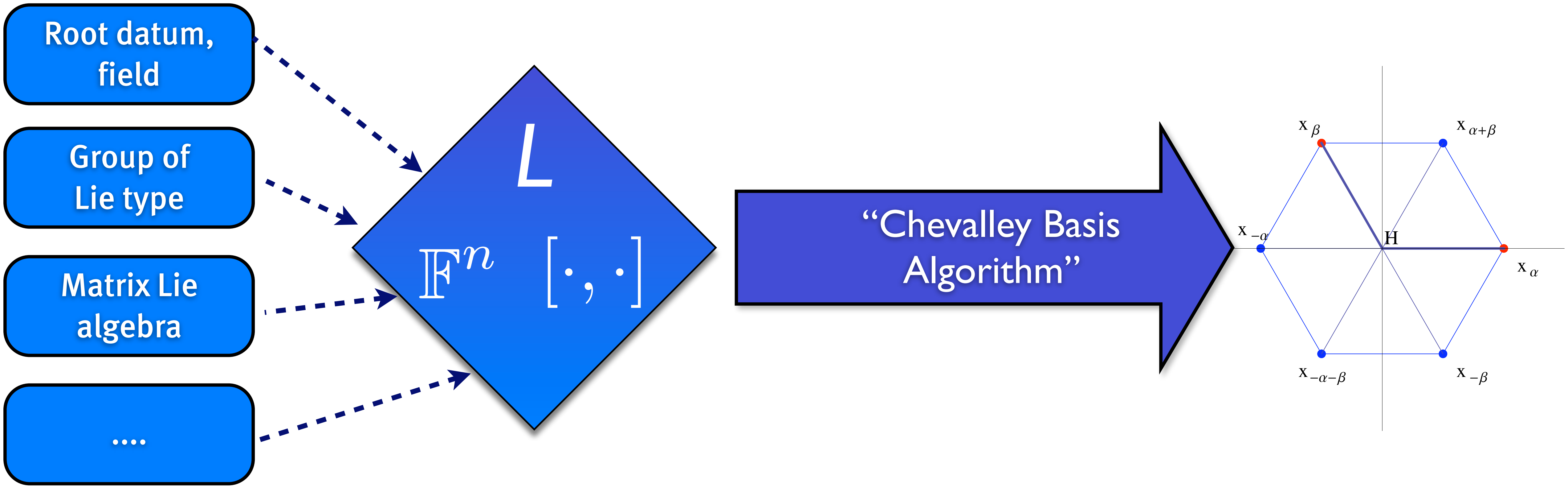
- **Assume** ***splitting Cartan subalgebra*** **H is given (Cohen/Murray, indep. Ryba);**
- **Assume root datum R is given**

The Mission



- **Char. 0, p ≥ 5: De Graaf, Murray; implemented in GAP, MAGMA**

The Mission

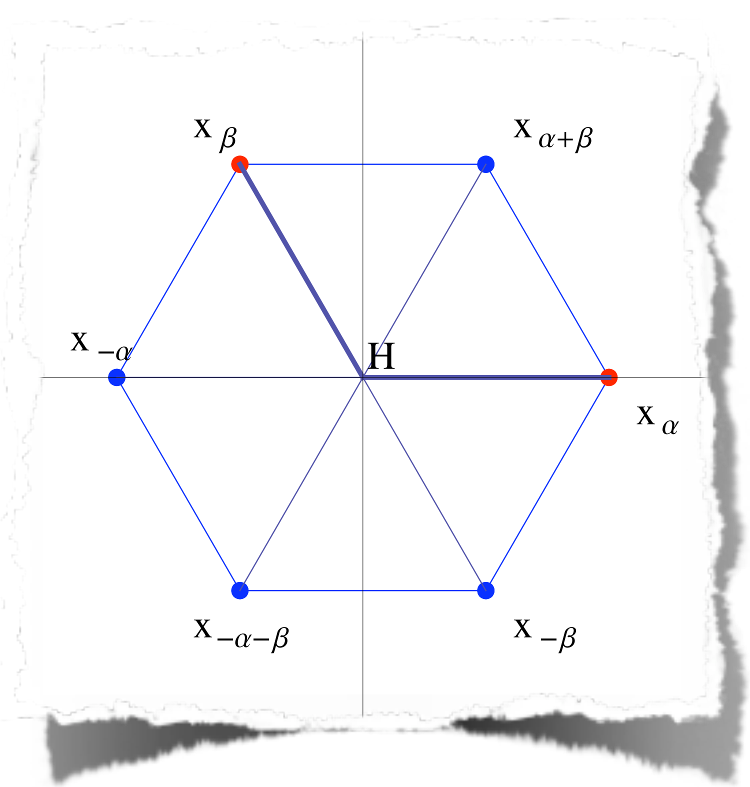


- **Char. 0, p ≥ 5: De Graaf, Murray; implemented in GAP, MAGMA**
- Char. 2,3: R., 2009, Implemented in MAGMA

The Problems

Normally:

- Diagonalise L using action of H on L (gives set of x_α),
- Use Cartan integers $\langle \alpha, \beta \rangle$ to "identify" the x_α ,
- Solve easy linear equations.



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and the Jacobi identity.

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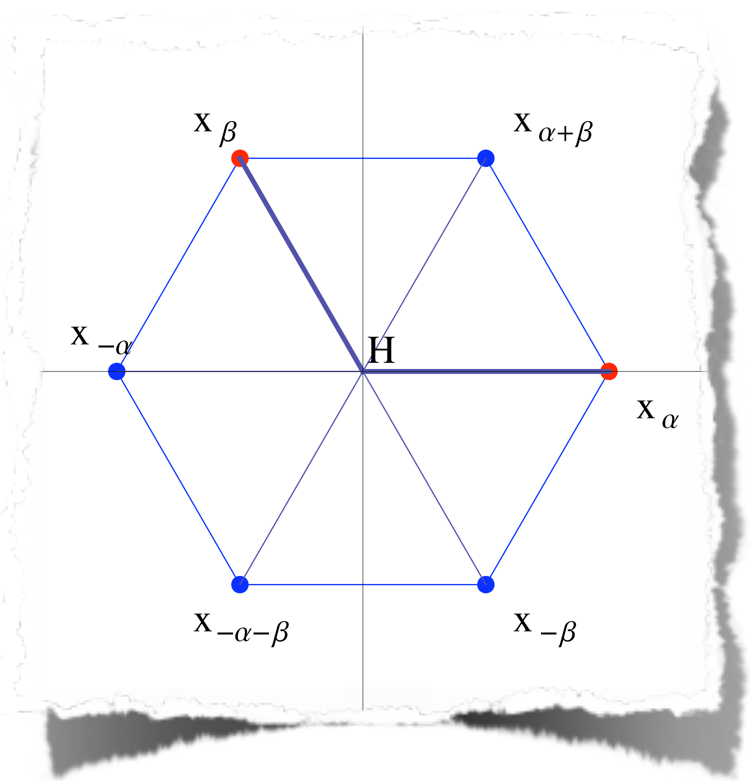

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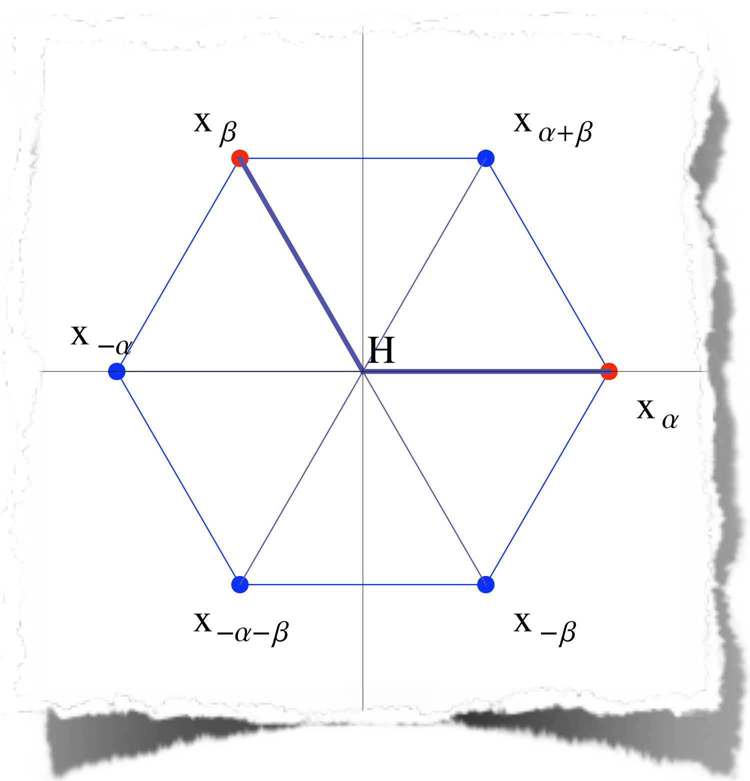

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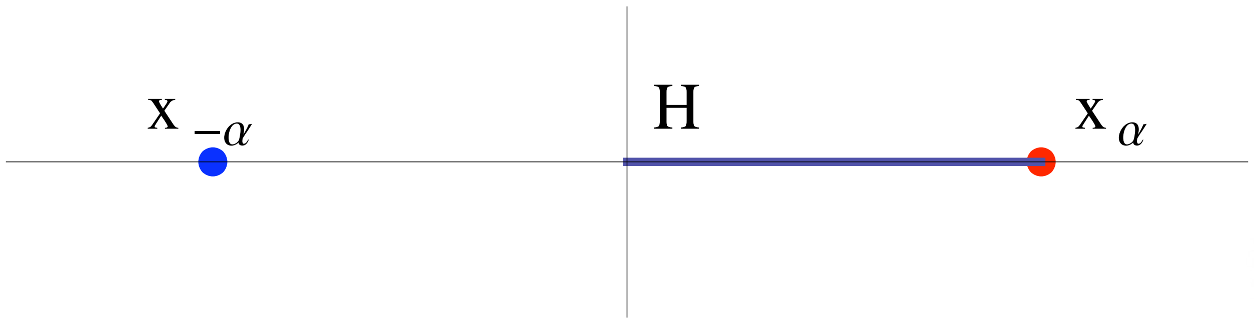
Diagonalising (A$_1$, char. 2)



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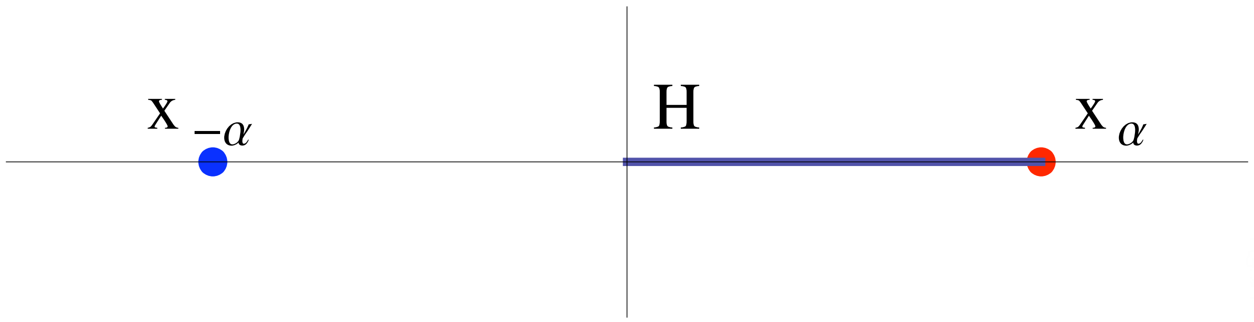


$$A_1^{\mathrm{Ad}} : X = Y = \mathbb{Z}$$

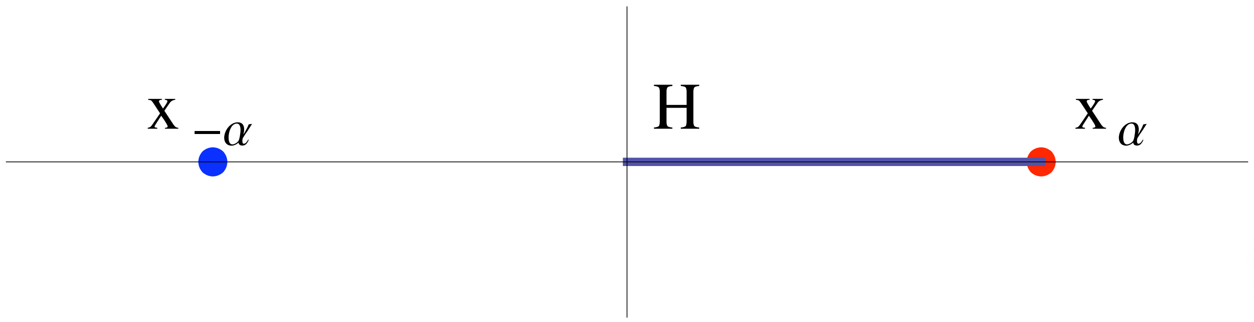
$$\Phi = \{\alpha = 1, -\alpha = -1\},$$

$$\Phi^\vee = \{\alpha^\vee = 2, -\alpha^\vee = -2\},$$

$$\begin{aligned}
[h_i, h_j] &= 0, \\
[x_\alpha, h_i] &= \langle \alpha, f_i \rangle x_\alpha, \\
[x_{-\alpha}, x_\alpha] &= \textstyle\sum_{i=1}^n \langle e_i, \alpha^\vee \rangle h_i, \\
[x_\alpha, x_\beta] &= \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and the Jacobi identity.

Diagonalising (A_1, char. 2)



$$A_1^{\mathrm{Ad}} : X = Y = \mathbb{Z}$$

$$\Phi = \{\alpha = 1, -\alpha = -1\},$$

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$$L = \mathbb{F}h \oplus \mathbb{F}x_\alpha \oplus \mathbb{F}x_{-\alpha}$$

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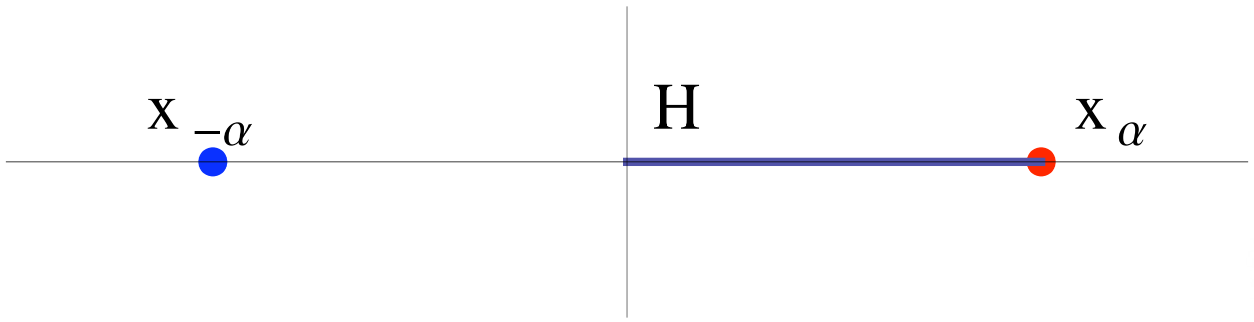

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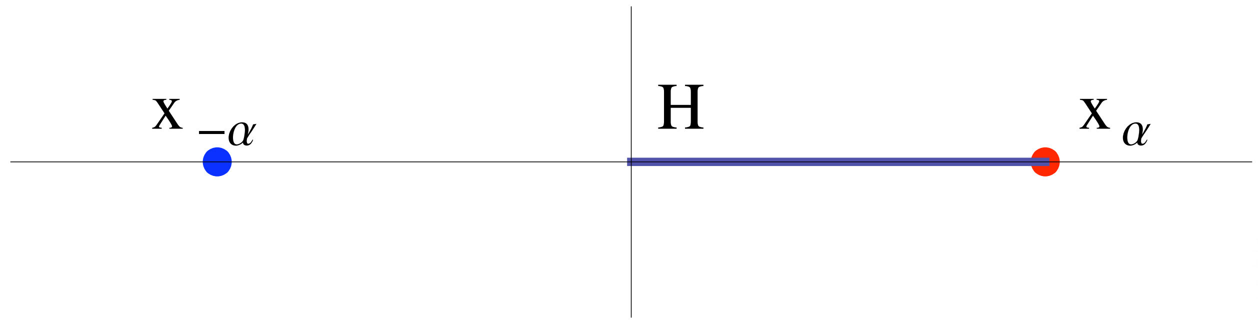
	x_α	$x_{-\alpha}$	h
x_α	0	$\langle e_1, \alpha^\vee\rangle h$	$\langle \alpha, f_1\rangle x_\alpha$
$x_{-\alpha}$		0	$\langle -\alpha, f_1\rangle x_{-\alpha}$
h			0

$$\begin{aligned}
[h_i, h_j] &= 0,\\
[x_\alpha, h_i] &= \langle \alpha, f_i\rangle x_\alpha,\\
[x_{-\alpha}, x_\alpha] &= \textstyle\sum_{i=1}^n \langle e_i, \alpha^\vee\rangle h_i,\\
[x_\alpha, x_\beta] &= \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

and the Jacobi identity.

Diagonalising (A₁, char. 2)



$$\mathrm{A}_1^{\mathrm{Ad}} : X = Y = \mathbb{Z}$$

$$\Phi = \{\alpha = 1, -\alpha = -1\},$$

$$\Phi^\vee = \{\alpha^\vee = 2, -\alpha^\vee = -2\},$$

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and the Jacobi identity.

	x_α	$x_{-\alpha}$	h
x_α	0	$\langle e_1, \alpha^\vee \rangle h$	$\langle \alpha, f_1 \rangle x_\alpha$
$x_{-\alpha}$		0	$\langle -\alpha, f_1 \rangle x_{-\alpha}$
h			0

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Basis transformation....

$$x = x_\alpha - x_{-\alpha}$$
$$y = 2x_\alpha + x_{-\alpha}$$

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Basis transformation....

$$\begin{aligned} x &= x_\alpha - x_{-\alpha} \\ y &= 2x_\alpha + x_{-\alpha} \end{aligned}$$

	x	y	h
x	0	$-6h$	$-\frac{1}{3}x + \frac{2}{3}y$
y	$6h$	0	$\frac{4}{3}x + \frac{1}{3}y$
h	$\frac{1}{3}x - \frac{2}{3}y$	$-\frac{4}{3}x - \frac{1}{3}y$	0

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Basis transformation....

$$x = x_\alpha - x_{-\alpha}$$
$$y = 2x_\alpha + x_{-\alpha}$$

	x	y	h
x	0	$-6h$	$-\frac{1}{3}x + \frac{2}{3}y$
y	$6h$	0	$\frac{4}{3}x + \frac{1}{3}y$
h	$\frac{1}{3}x - \frac{2}{3}y$	$-\frac{4}{3}x - \frac{1}{3}y$	0

Algorithm:

- **Diagonalize L wrt H**
- **Find 1-dim eigenspaces:**

$$S_1, S_{-1}, S_0$$

- **Take**

$$x + y \in S_1$$
$$x - \frac{1}{2}y \in S_{-1}$$
$$h \in S_0$$

- **Done!**

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Diagonalising (A_1, char. 2)

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Diagonalising (A_1, char. 2)

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$$x + y \in S_1$$
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$$h \in S_0$$

- Done!

But in char. 2...

- Diagonalize L wrt H
- Find 1-dim eigenspace: S_0
- Find 2-dim eigenspace: S_1
- ...

Diagonalising (A_1, char. 2)

Algorithm:

- Diagonalize L wrt H
- Find 1-dim eigenspaces:

$$S_1, S_{-1}, S_0$$

- Take

$$x + y \in S_1$$
$$x - \frac{1}{2}y \in S_{-1}$$
$$h \in S_0$$

- Done!

But in char. 2...

- Diagonalize L wrt H
- Find 1-dim eigenspace: S_0
- Find 2-dim eigenspace: S_1
- ...
- Not really an issue here (almost anything will do), but non-trivial in many other cases.

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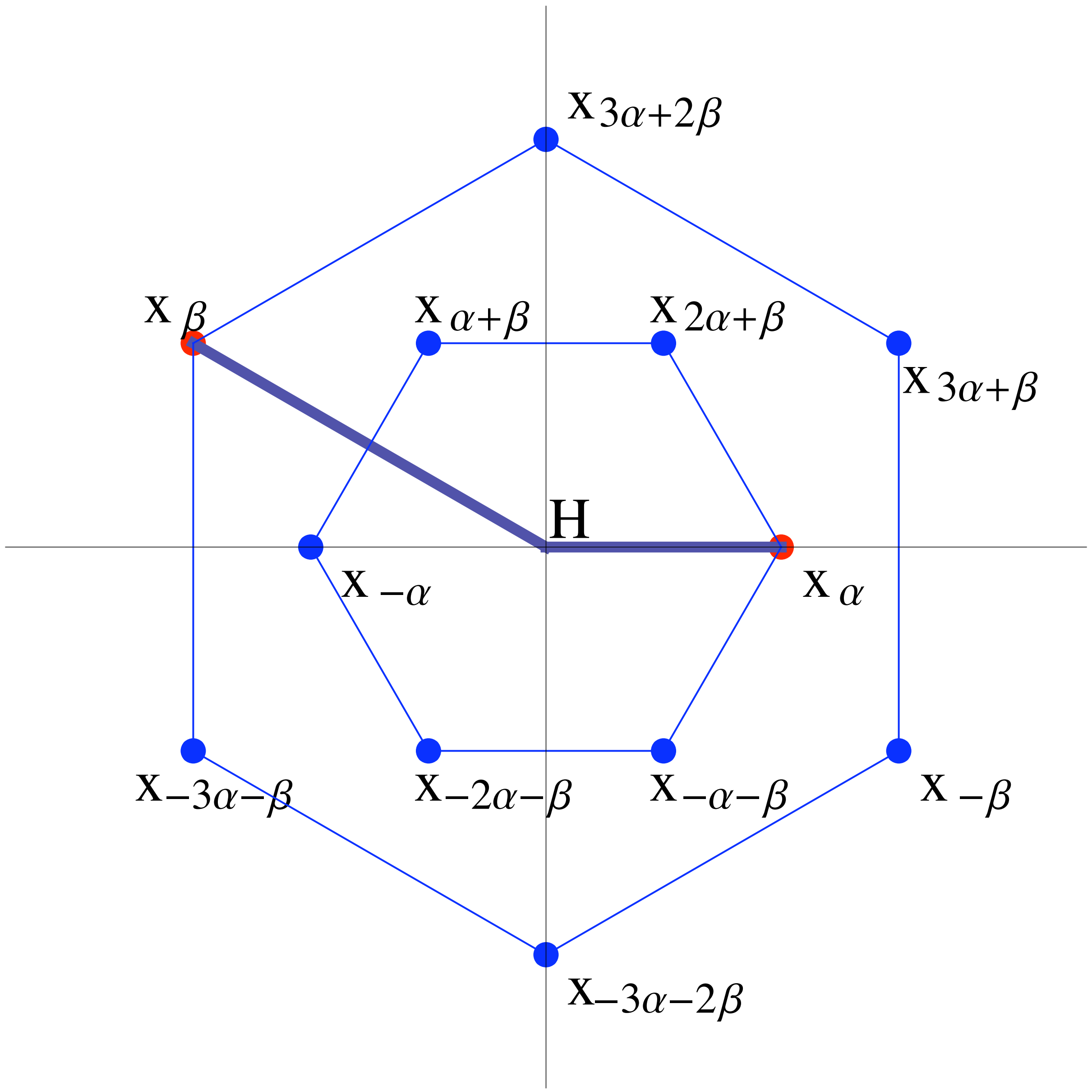
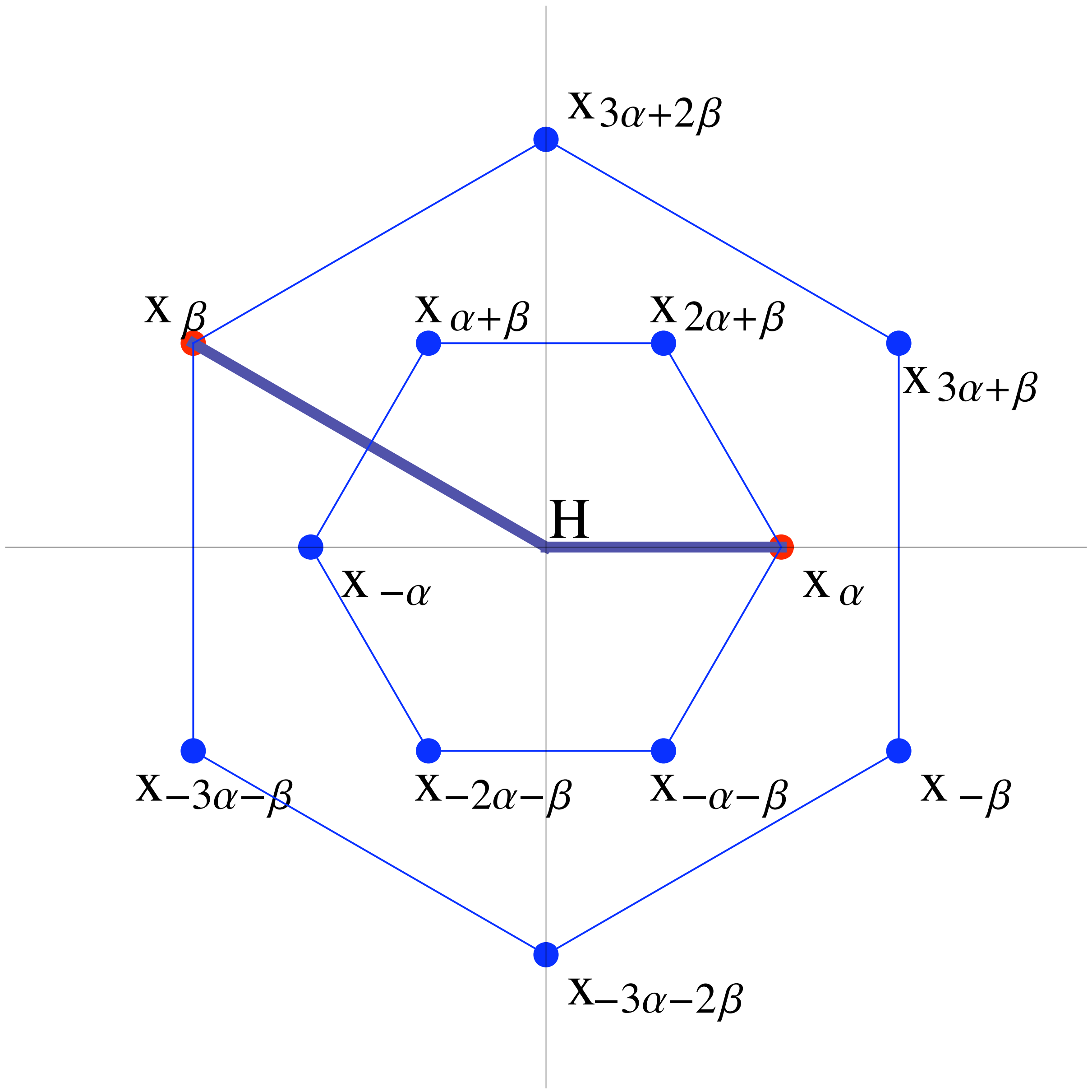
Diagonalising (G_2, char. 3)



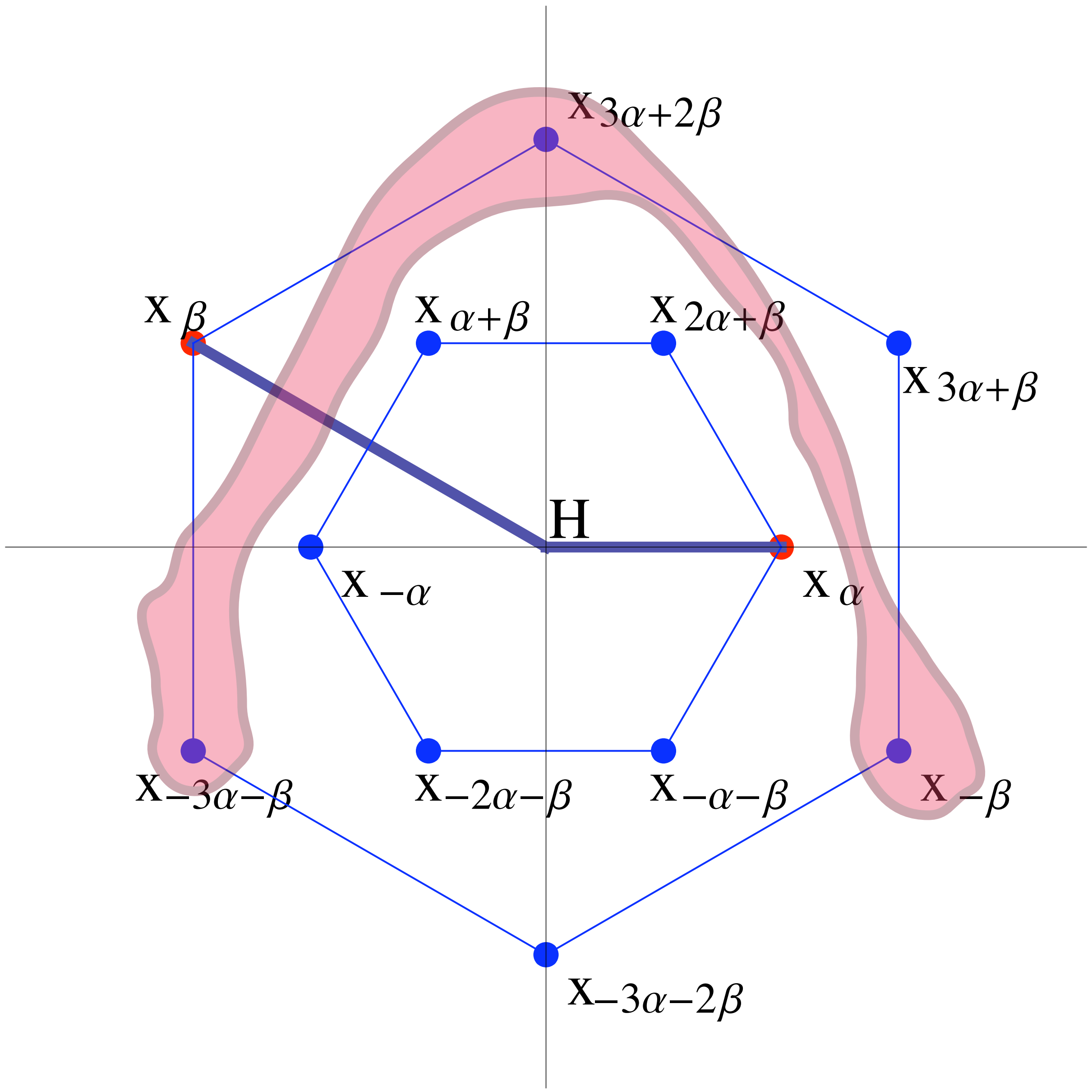
Diagonalising (G_2, char. 3)



In char. 3...

- **Find 1 2-dim eigenspace,**
- **Find 6 1-dim eigenspaces,**
- **Find 2 3-dim eigenspaces.**

Diagonalising (G_2, char. 3)



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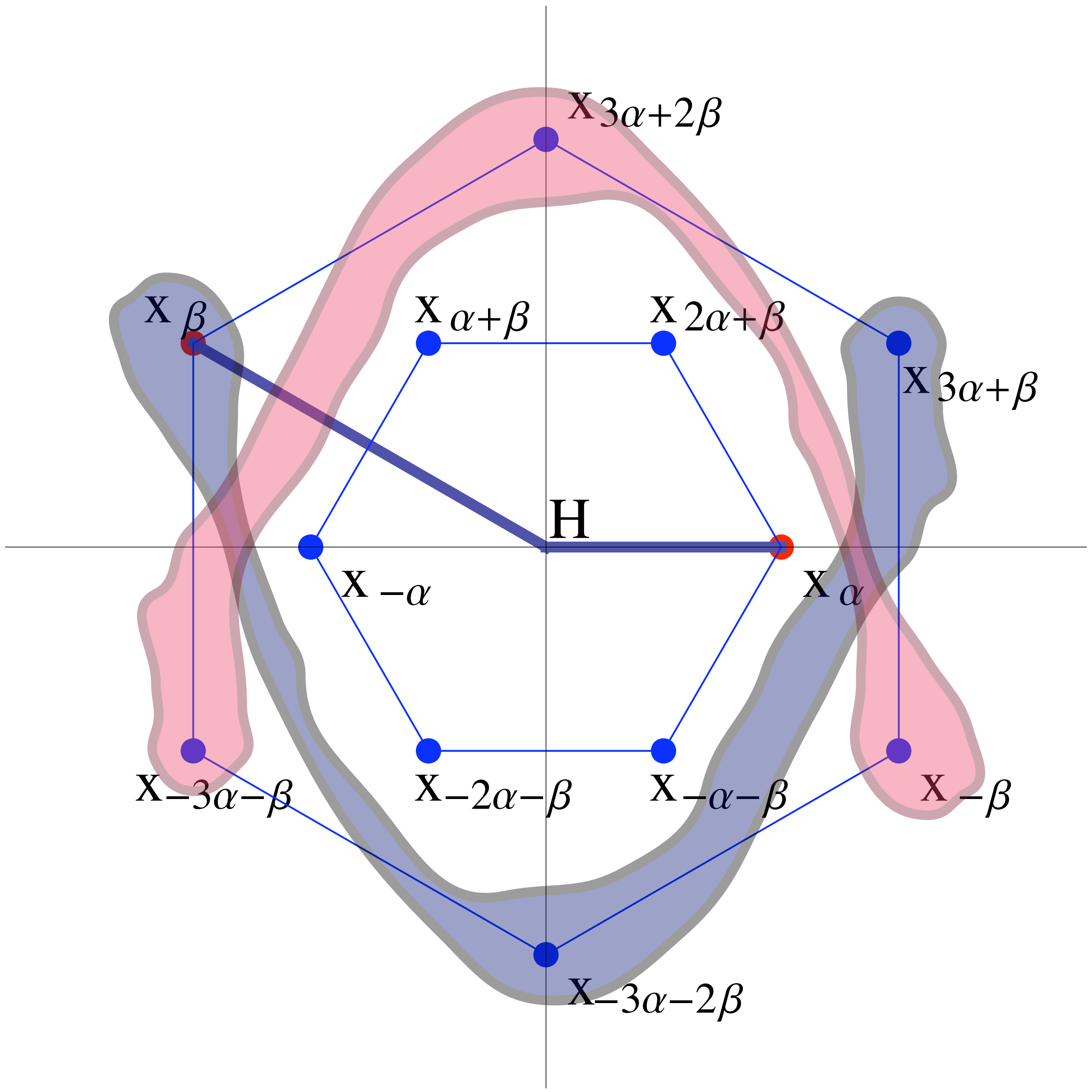

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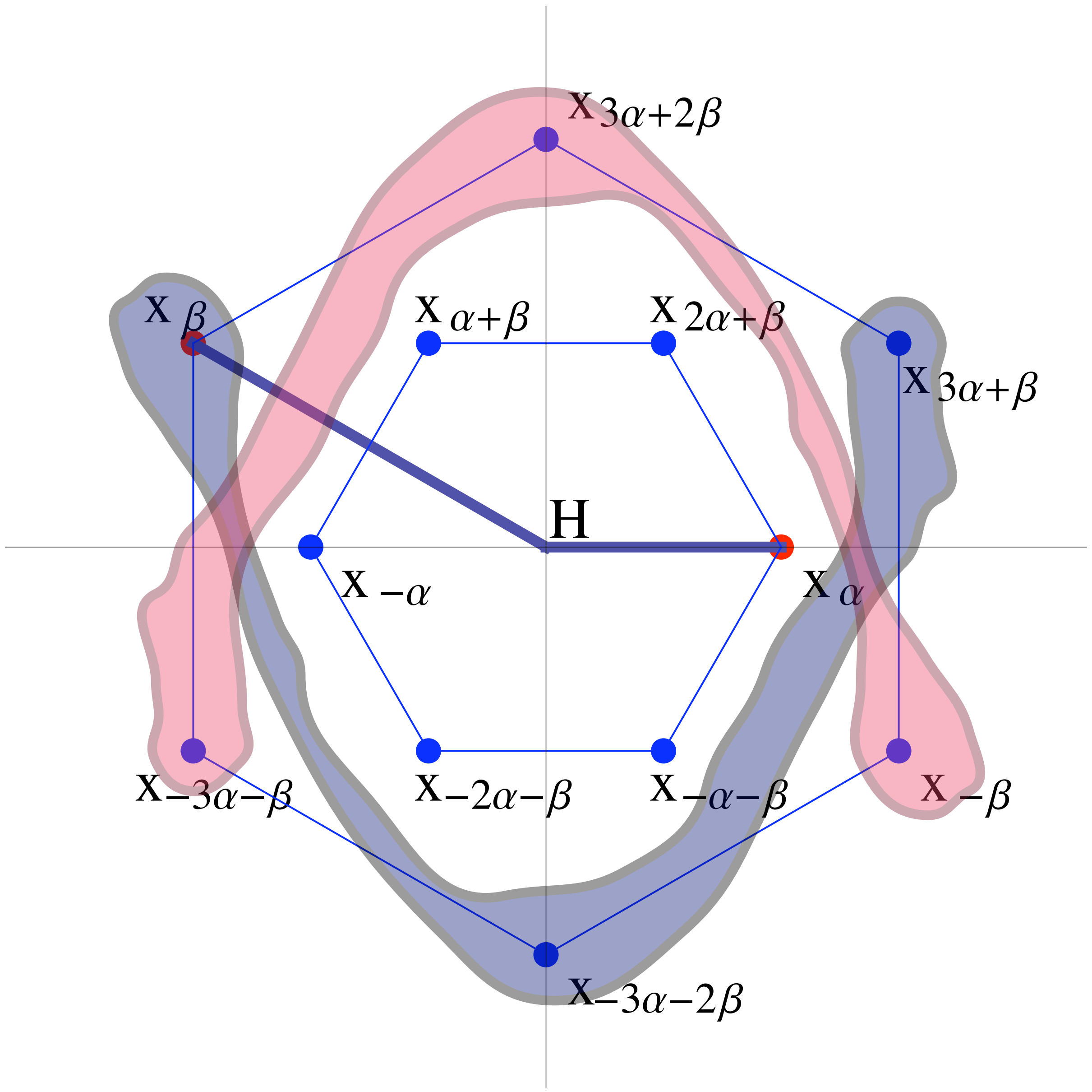

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Observe:

- $[\mathbb{F}x_\beta, \mathbb{F}x_{\pm(\alpha)}] = \mathbb{F}x_{\alpha+\beta}$
- $[\mathbb{F}x_\beta, \mathbb{F}x_{\pm(\alpha+\beta)}] = \mathbb{F}x_{-\alpha}$
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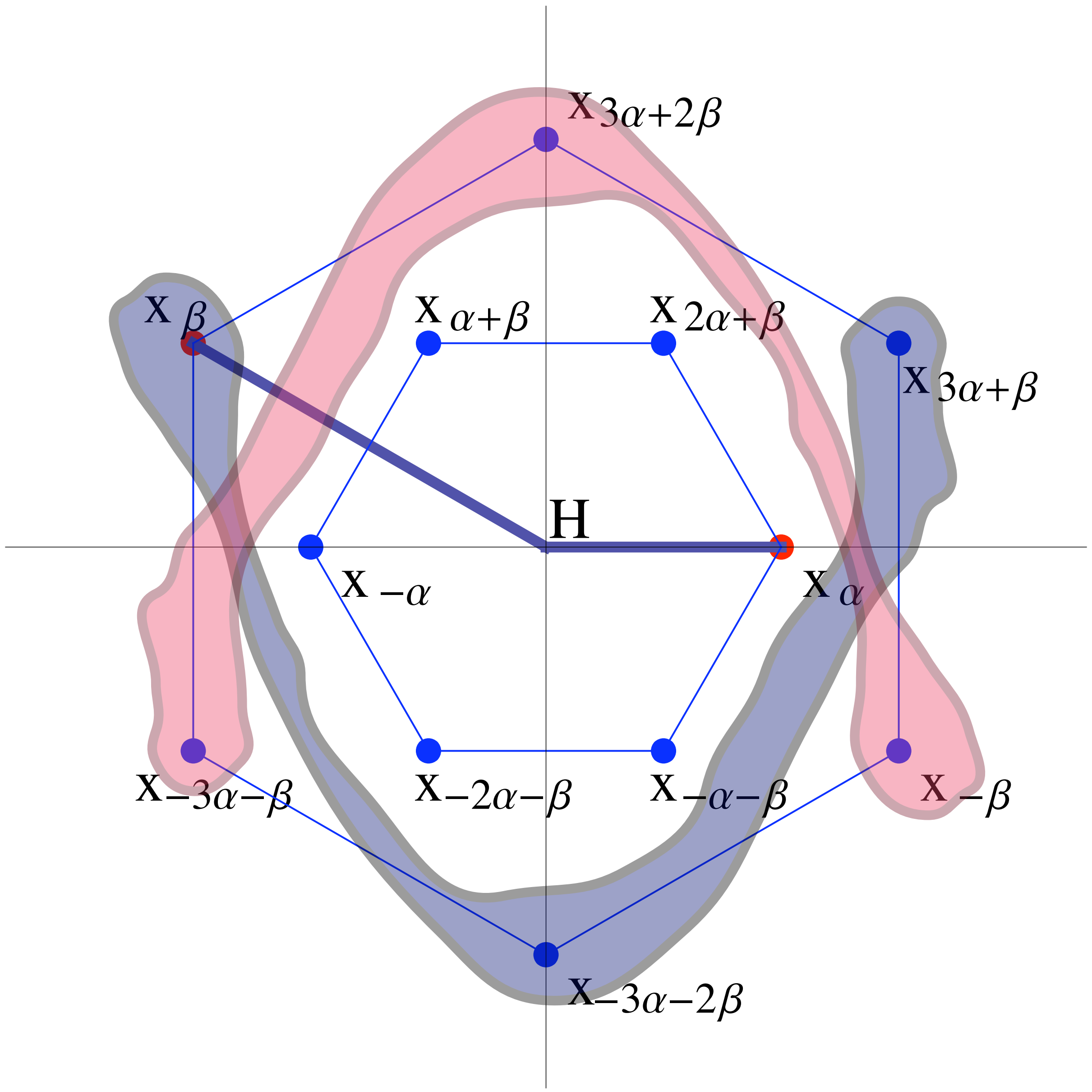
Diagonalising (G_2, char. 3)



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- $[\mathbb{F}x_\beta, \mathbb{F}x_{\pm(2\alpha+\beta)}] = 0$

So find root spaces in 3-dim S:

- For $\gamma \in \{\alpha, \alpha+\beta, 2\alpha+\beta\}$ compute $C_S(\mathbb{F}x_\gamma, \mathbb{F}x_{-\gamma})$

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Diagonalising (overview)

$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$\mathrm{A_2}^{\mathrm{sc}}(3)$	3^2	[Der]	$\mathrm{C}_n{}^{\mathrm{ad}}(2)$ $(n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$\mathrm{G_2}(3)$	$1^6, 3^2$	[C]	$\mathrm{C}_n{}^{\mathrm{sc}}(2)$ $(n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	$[\mathrm{B_2}^{\mathrm{sc}}]$
$\mathrm{A}_3^{\mathrm{sc},(2)}(2)$	4^3	[Der]	$\mathrm{D}_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$\mathrm{B_2}^{\mathrm{ad}}(2)$	$2^2, 4$	[C]	$\mathrm{D_4}^{\mathrm{sc}}(2)$	8^3	[Der]
$\mathrm{B}_n{}^{\mathrm{ad}}(2)$ $(n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$\mathrm{D}_n^{(1)}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_2}^{\mathrm{sc}}(2)$	$\mathbf{4}, 4$	$[\mathrm{B_2}^{\mathrm{sc}}]$	$\mathrm{D}_n{}^{\mathrm{sc}}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_3}^{\mathrm{sc}}(2)$	6^3	[Der]	$\mathrm{F_4}(2)$	$2^{12}, 8^3$	[C]
$\mathrm{B_4}^{\mathrm{sc}}(2)$	$2^4, 8^3$	[Der]	$\mathrm{G_2}(2)$	4^3	[Der]
$\mathrm{B}_n{}^{\mathrm{sc}}(2)$ $(n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	$[\mathrm{A_2}]$

TABLE 1. Multidimensional root spaces

Diagonalising (overview)

$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$A_2{}^{sc}(3)$	3^2	[Der]	$C_n{}^{ad}(2)$ $(n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$G_2(3)$	$1^6, 3^2$	[C]	$C_n{}^{sc}(2)$ $(n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	[$B_2{}^{sc}$]
$A_3^{sc,(2)}(2)$	4^3	[Der]	$D_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$B_2{}^{ad}(2)$	$2^2, 4$	[C]	$D_4{}^{sc}(2)$	8^3	[Der]
$B_n{}^{ad}(2)$ $(n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$D_n^{(1)}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$B_2{}^{sc}(2)$	$\mathbf{4}, 4$	[$B_2{}^{sc}$]	$D_n{}^{sc}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$B_3{}^{sc}(2)$	6^3	[Der]	$F_4(2)$	$2^{12}, 8^3$	[C]
$B_4{}^{sc}(2)$	$2^4, 8^3$	[Der]	$G_2(2)$	4^3	[Der]
$B_n{}^{sc}(2)$ $(n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	[A_2]

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$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$A_2{}^{sc}(3)$	3^2	[Der]	$C_n{}^{ad}(2)$ $(n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$G_2(3)$	$1^6, 3^2$	[C]	$C_n{}^{sc}(2)$ $(n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	[$B_2{}^{sc}$]
$A_3^{sc,(2)}(2)$	4^3	[Der]	$D_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$B_2{}^{ad}(2)$	$2^2, 4$	[C]	$D_4{}^{sc}(2)$	8^3	[Der]
$B_n{}^{ad}(2)$ $(n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$D_n^{(1)}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$B_2{}^{sc}(2)$	$\mathbf{4}, 4$	[$B_2{}^{sc}$]	$D_n{}^{sc}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
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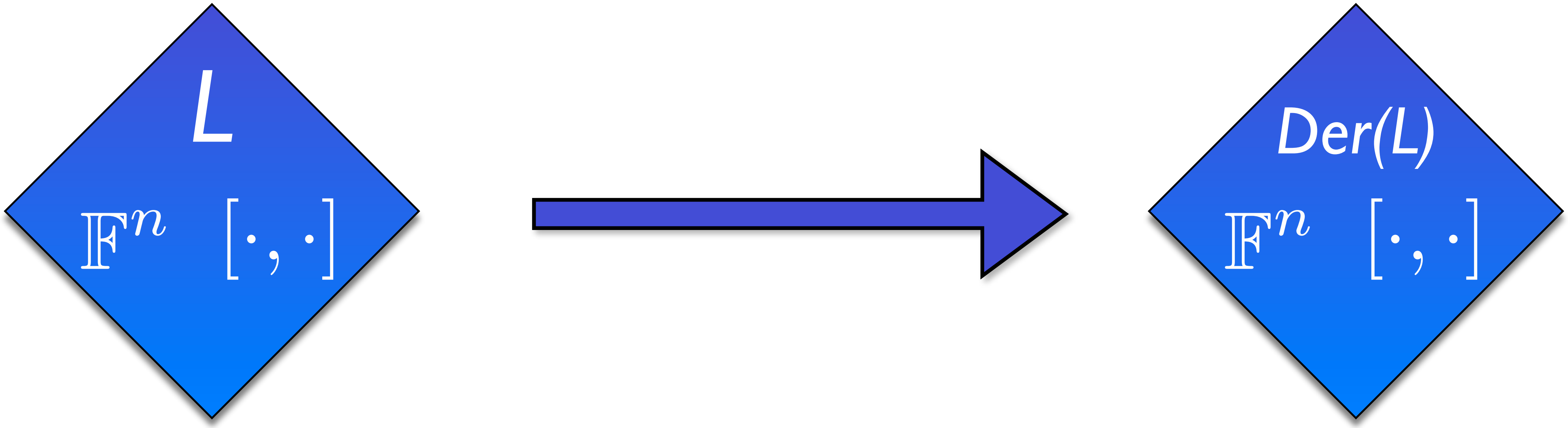
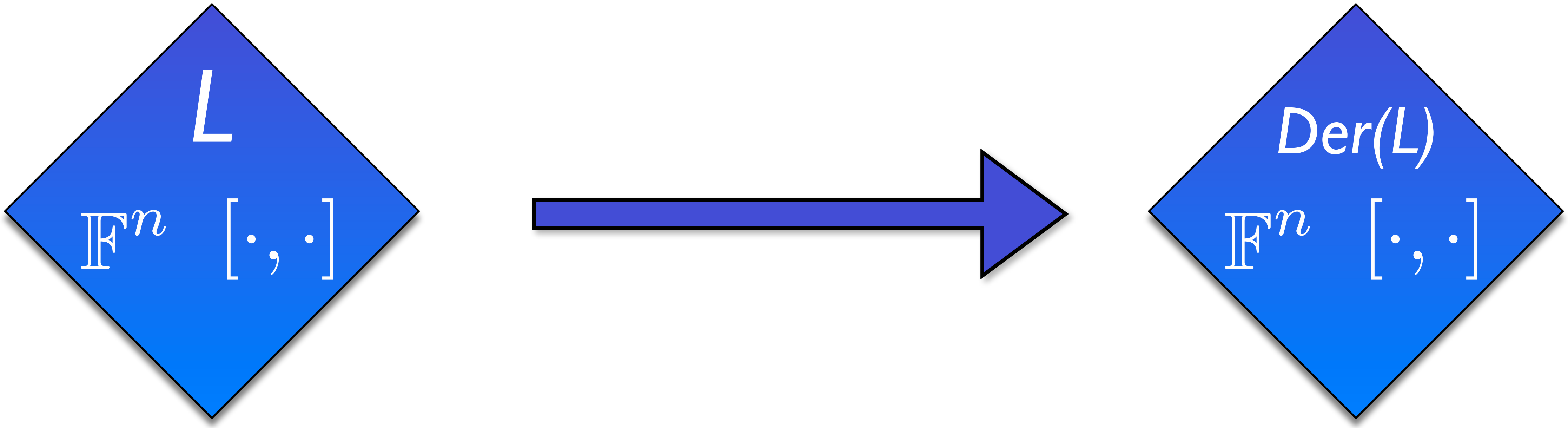
$B_n{}^{sc}(2)$ $(n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	[A_2]

TABLE 1. Multidimensional root spaces

Derivation (Lie) Algebra

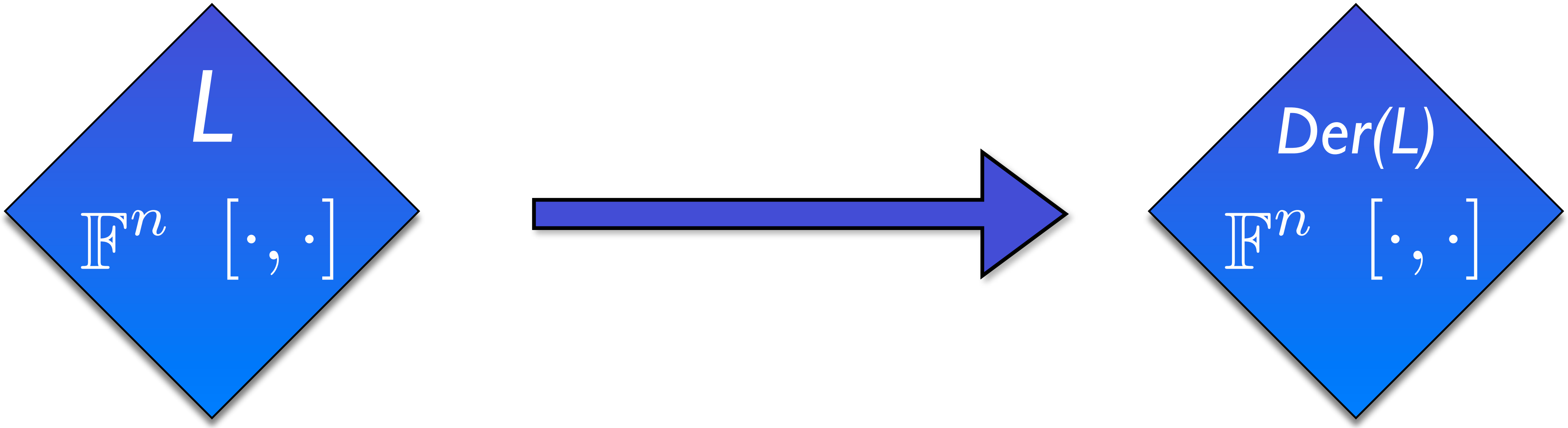


Derivation (Lie) Algebra



- $\mathrm{Der}(L) = \{d \in \mathrm{End}(L) \mid d([x,y]) = [d(x),y] + [x,d(y)]\}$

Derivation (Lie) Algebra



- $\mathrm{Der}(L) = \{d \in \mathrm{End}(L) \mid d([x,y]) = [d(x),y] + [x,d(y)]\}$
- **Observe:**
  - $\mathrm{Der}(L)$ **is a Lie algebra**
  - **(almost)** $L \subseteq \mathrm{Der}(L)$

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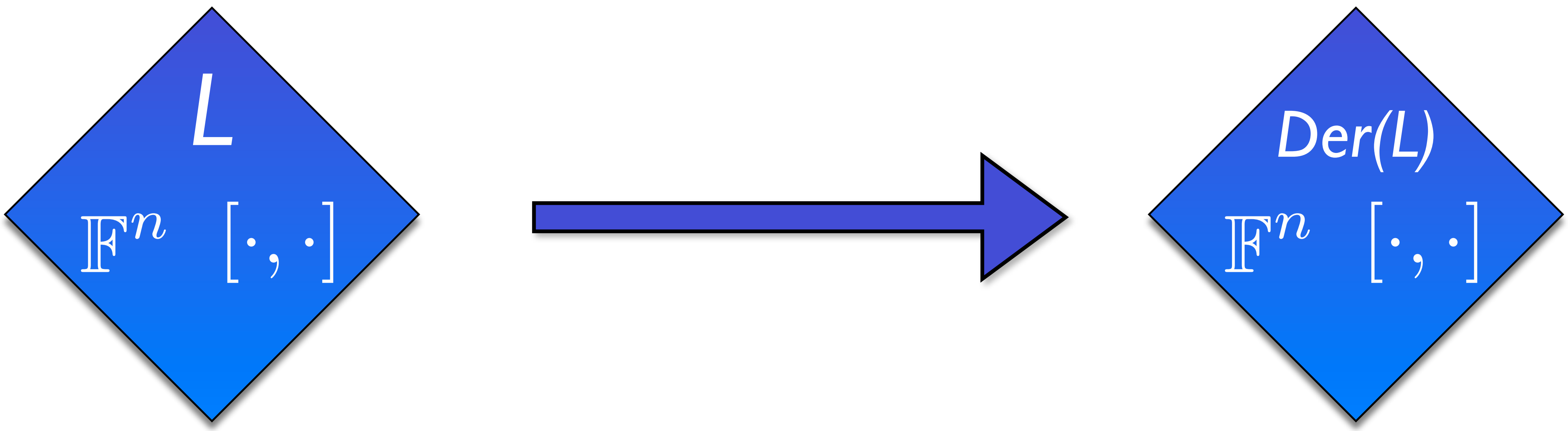
Derivation (Lie) Algebra

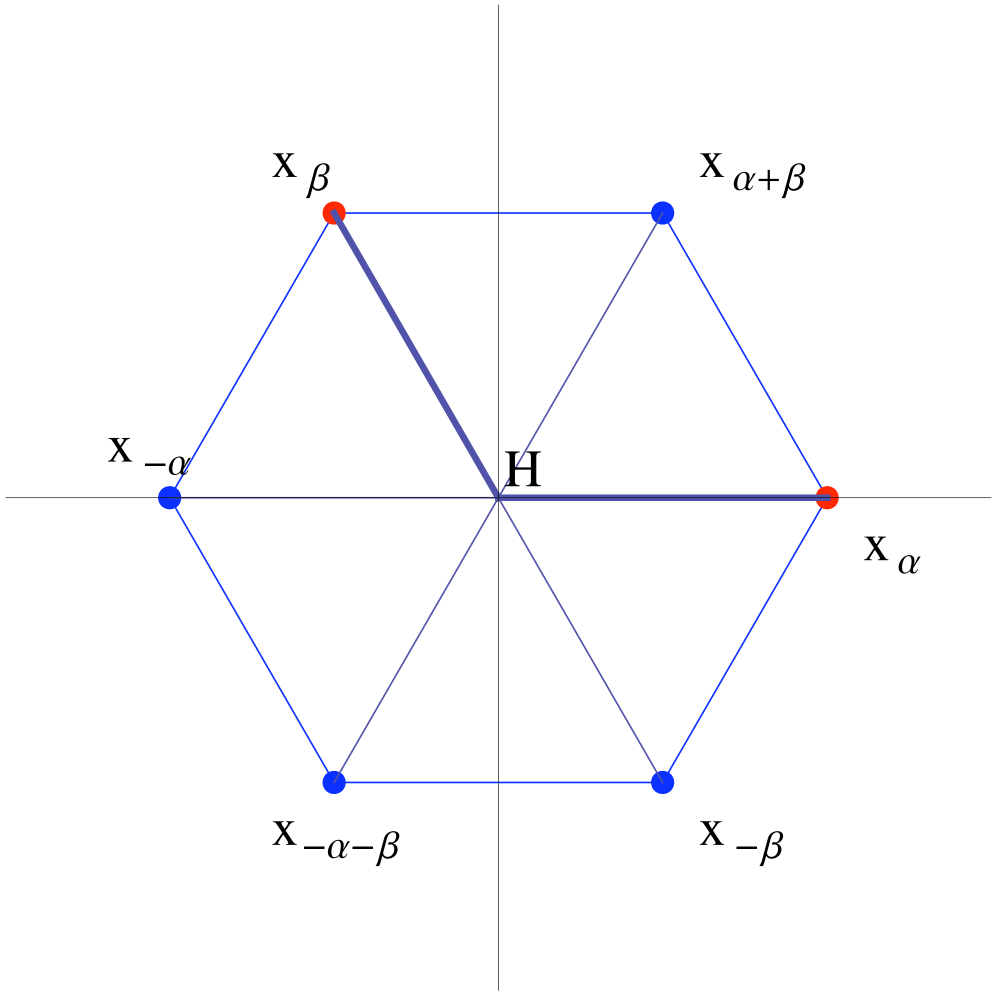


- $\mathrm{Der}(L) = \{d \in \mathrm{End}(L) \mid d([x,y]) =$ [...]
- **Observe:**
  - $\mathrm{Der}(L)$ **is a Lie algebra**
  - **(almost)** $L \subseteq \mathrm{Der}(L)$

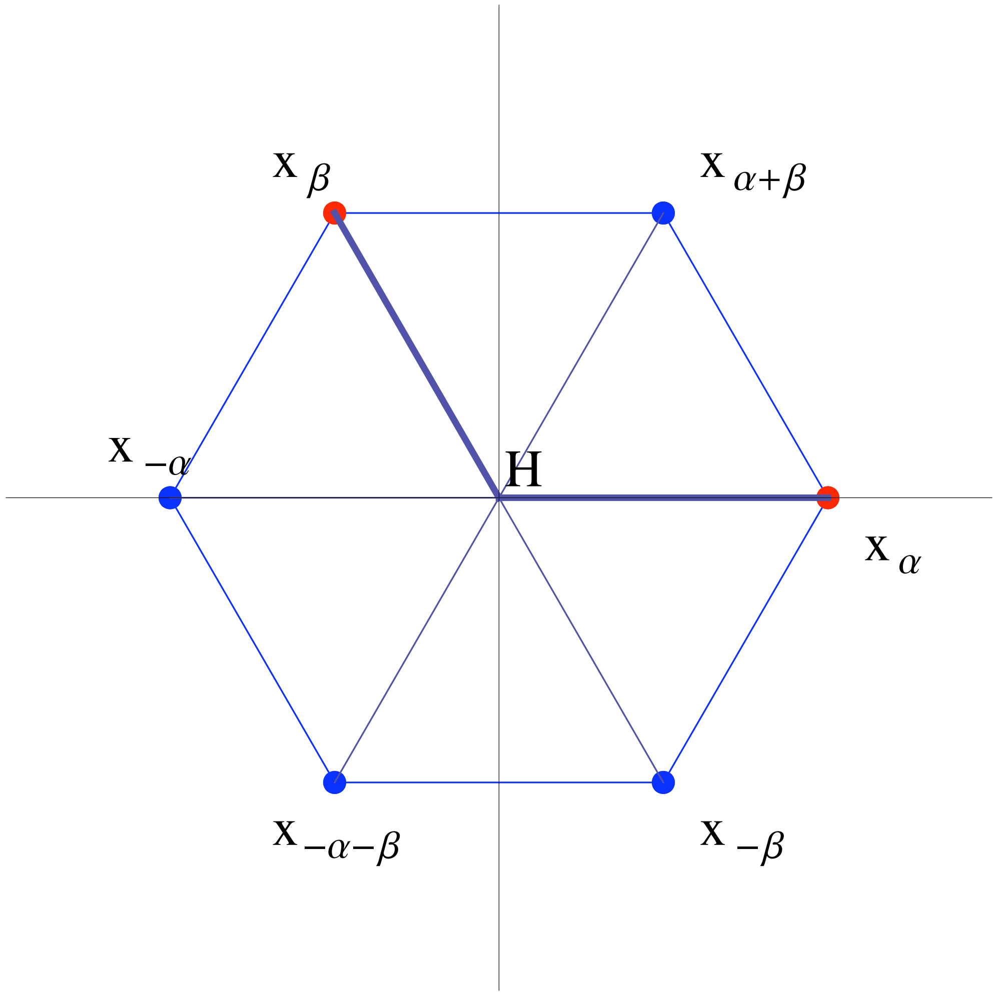
$$\begin{aligned}
\mathrm{ad}_z([x,y]) &= [z,[x,y]] \\
&= -[x,[y,z]] - [y,[z,x]] \\
&= [x,[z,y]] + [[z,x],y] \\
&= [x,\mathrm{ad}_z(y)] + [\mathrm{ad}_z(x),y]
\end{aligned}$$

Diagonalising (A_2, char. 3)

Adjoint



Simply Connected



Diagonalising (A_2, char. 3)

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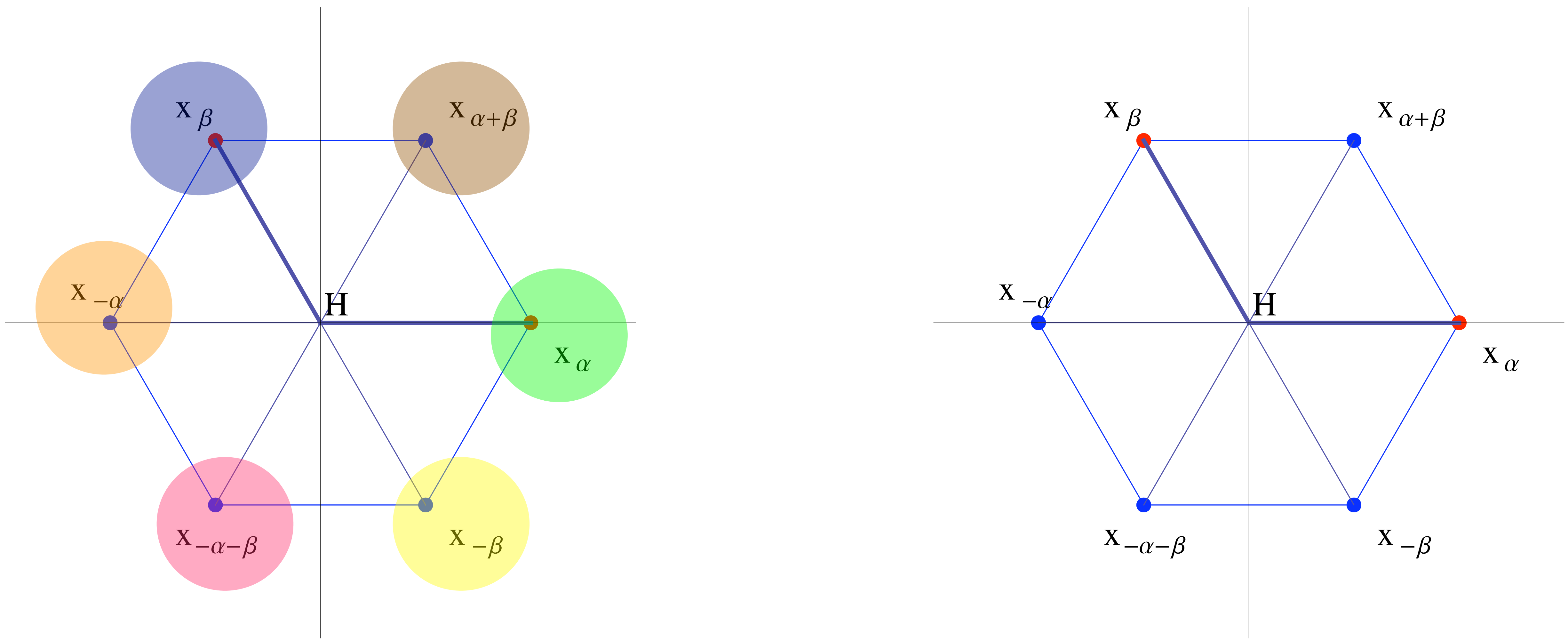
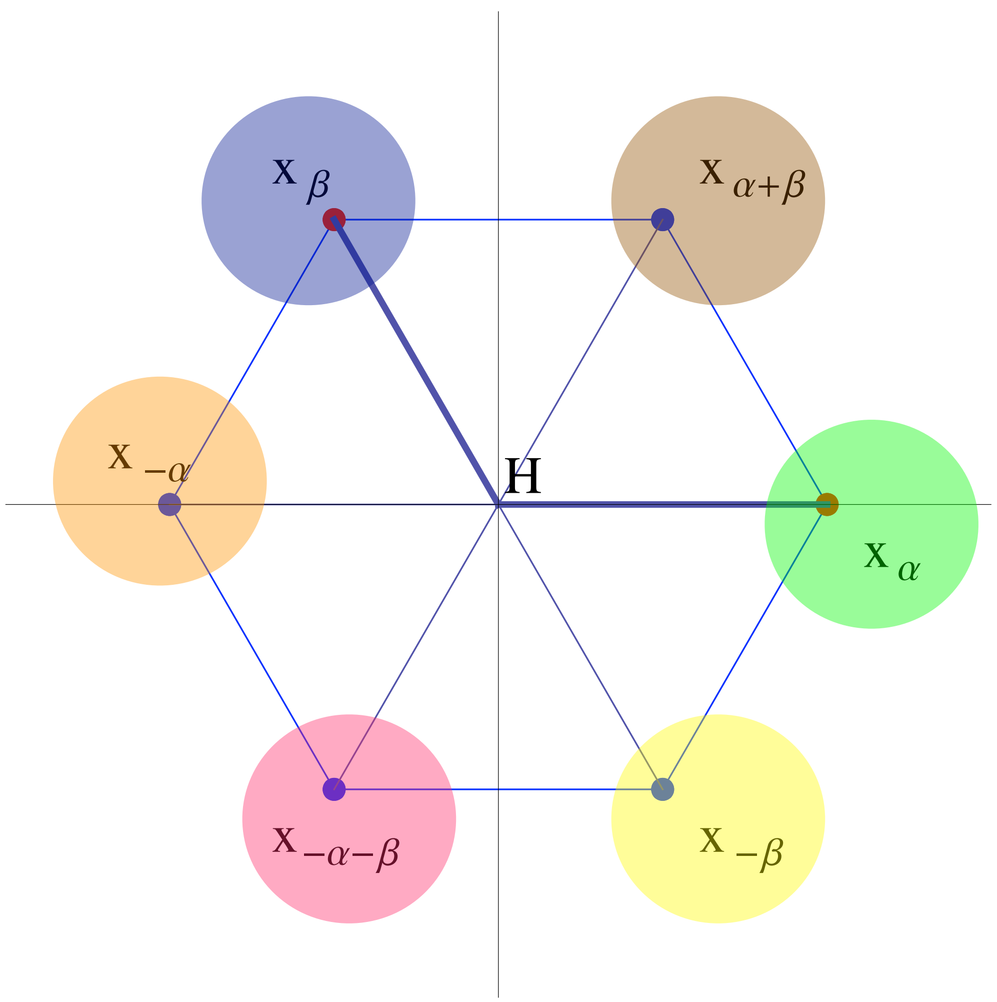


6 one-dimensional spaces

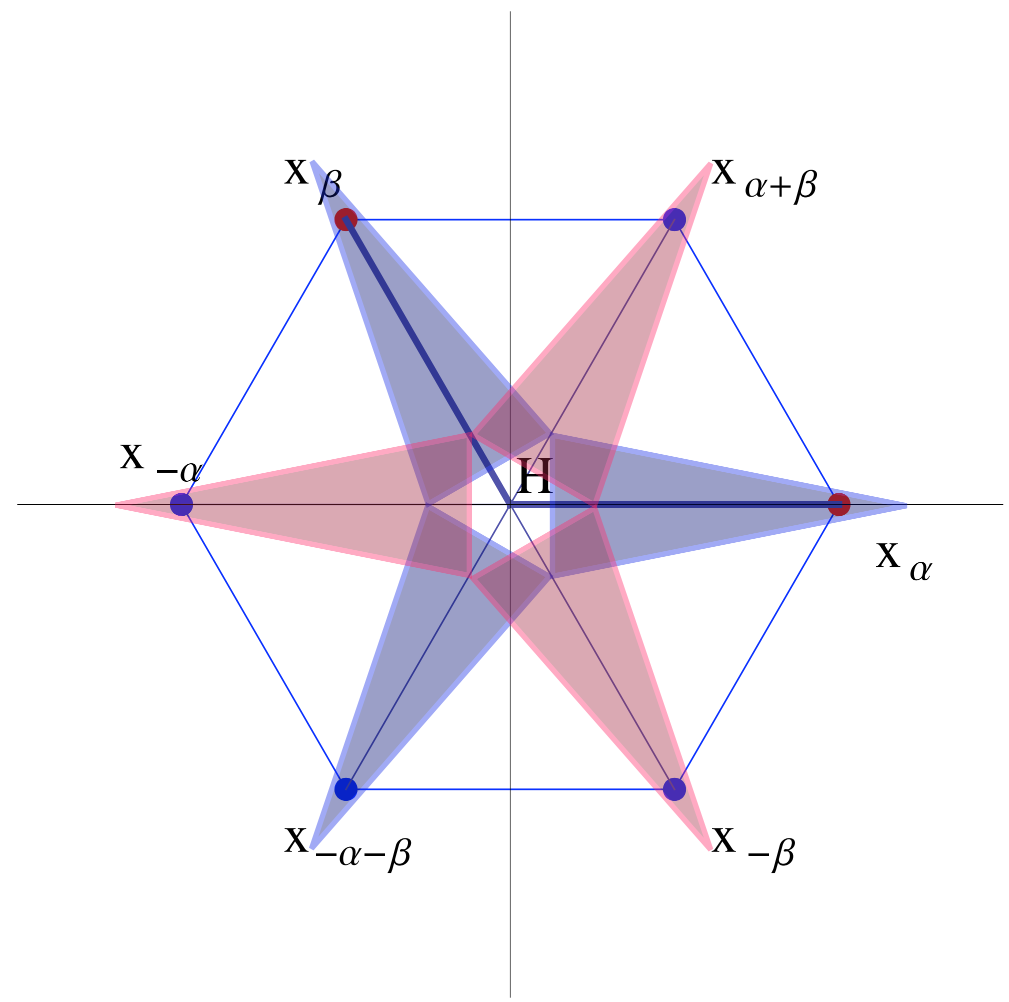
Diagonalising (A_2, char. 3)

Adjoint



6 one-dimensional spaces

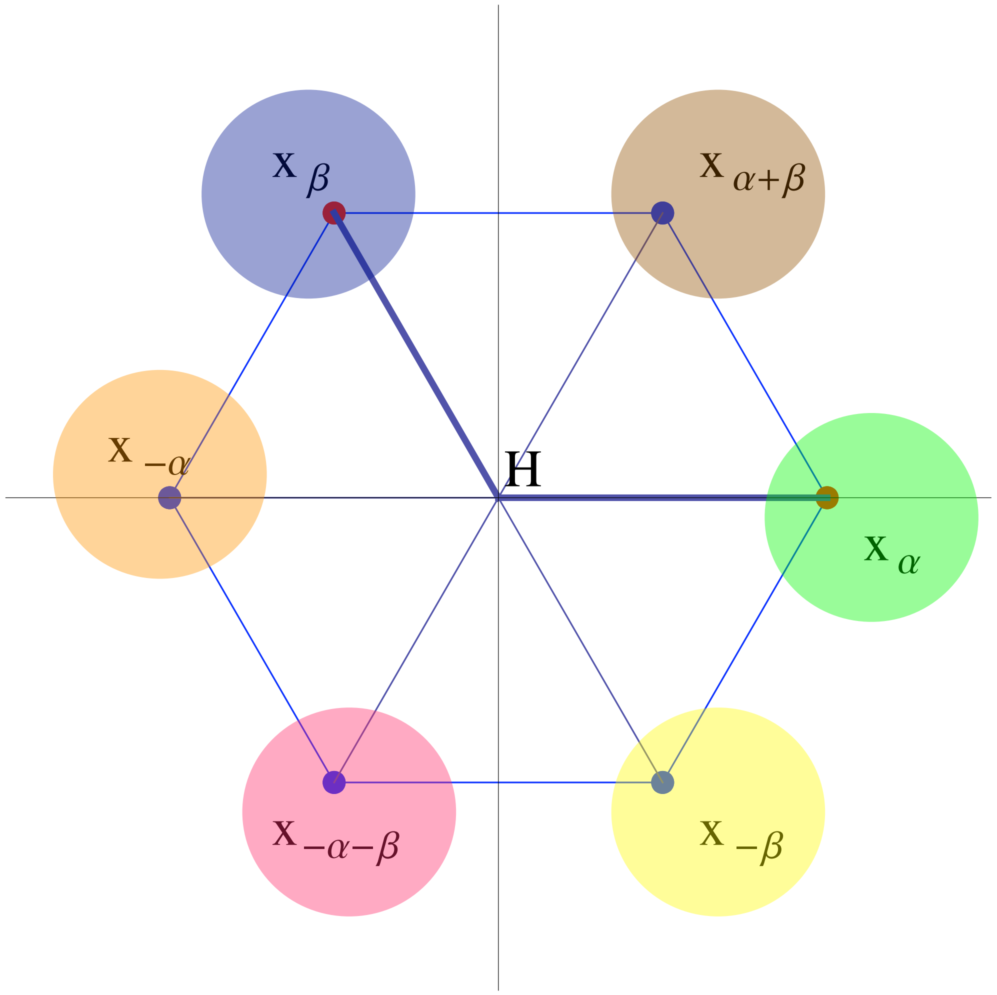
Simply Connected

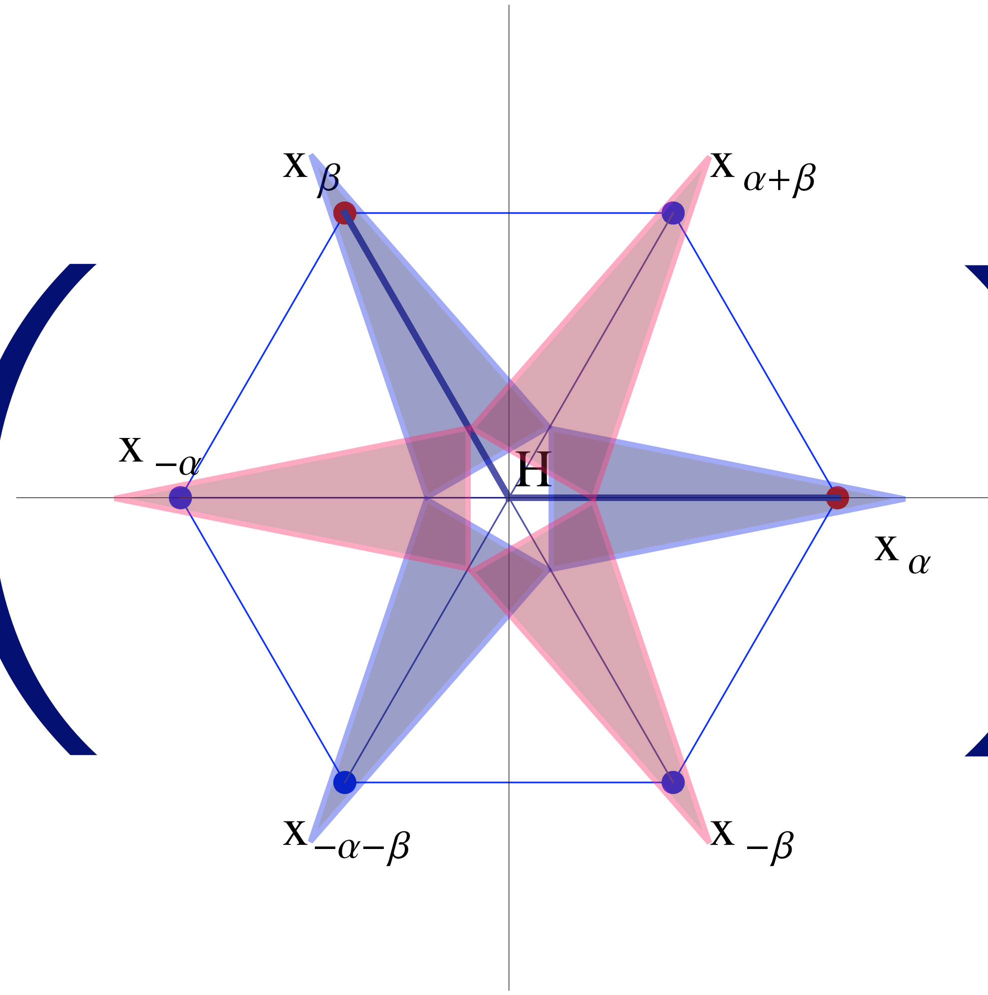


2 three-dimensional spaces

Diagonalising (A_2, char. 3)

Adjoint | **Simply Connected**



= Der ()!

6 one-dimensional spaces | 2 three-dimensional spaces

Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- **What next?**

Conclusion

- **Main challenges for computing Chevalley bases in small characteristic:**
  - **Multidimensional eigenspaces,**
  - **Broken root chains;**

Conclusion

- Main challenges for computing Chevalley bases in small characteristic:
  - Multidimensional eigenspaces,
  - Broken root chains;
- Found solutions for all cases, and implemented these:
  - MAGMA package, about 6000 lines,
  - soon to be available.

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  - Compute split Cartan subalgebras in small characteristic;

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Conclusion

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  - Broken root chains;
- Found solutions for all cases, and implemented these:
  - MAGMA package, about 6000 lines,
  - soon to be available.
- To do:
  - Compute split Cartan subalgebras in small characteristic;
- Bigger picture:
  - Recognition of groups or Lie algebras,
  - Finding conjugators for Lie group elements,
  - Finding automorphisms of Lie algebras,
  - ...

Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- What next?

- **Questions?**

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