Construction of Chevalley Bases of Lie Algebras

Dan Roozemond

Joint work with Arjeh M. Cohen

May 14th, 2009, Computeralgebratagung, Universität Kassel

http://www.win.tue.nl/~droozemo/

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Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- Does it work?
- What next?

What is a Lie Algebra?

- **Vector space:** $\mathbb{F}^n$

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- **Multiplication** $[\cdot,\cdot] : L \times L \mapsto L$ **that is**
  - **Bilinear,**
  - **Anti-symmetric,**
  - **Satisfies Jacobi identity:**

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

What is a Lie Algebra?

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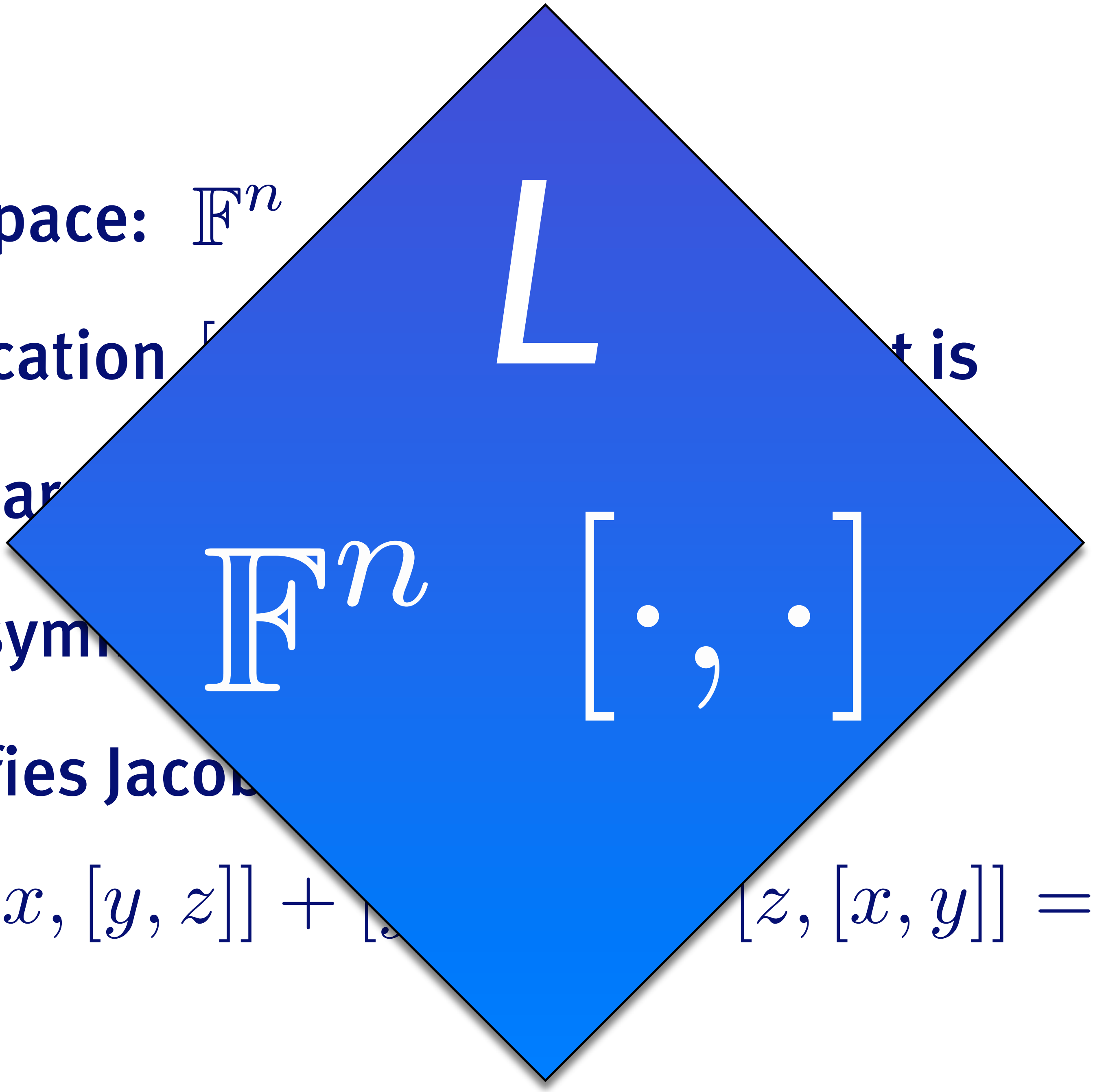
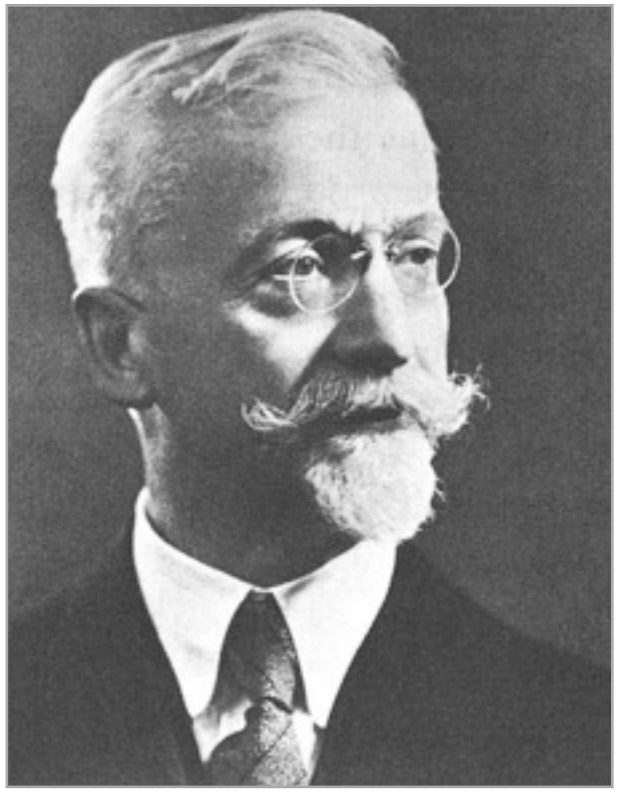


Simple Lie algebras



Classification (Killing, Cartan)

If $\mathrm{char}(\mathbb{F}) = 0$ or big enough then the only simple Lie algebras are:

$$\mathrm{A}_n \ (n \geq 1) \qquad \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$$

$$\mathrm{B}_n \ (n \geq 2) \qquad \mathrm{F}_4$$

$$\mathrm{C}_n \ (n \geq 3) \qquad \mathrm{G}_2$$

$$\mathrm{D}_n \ (n \geq 4)$$

Why Study Lie Algebras?

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- Study *groups* by their Lie algebras:
  - Simple algebraic group G <-> Unique Lie algebra L
  - Many properties carry over to L
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- Opportunities for:
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- Because there are problems to be solved!
  - ... and a thesis to be written...

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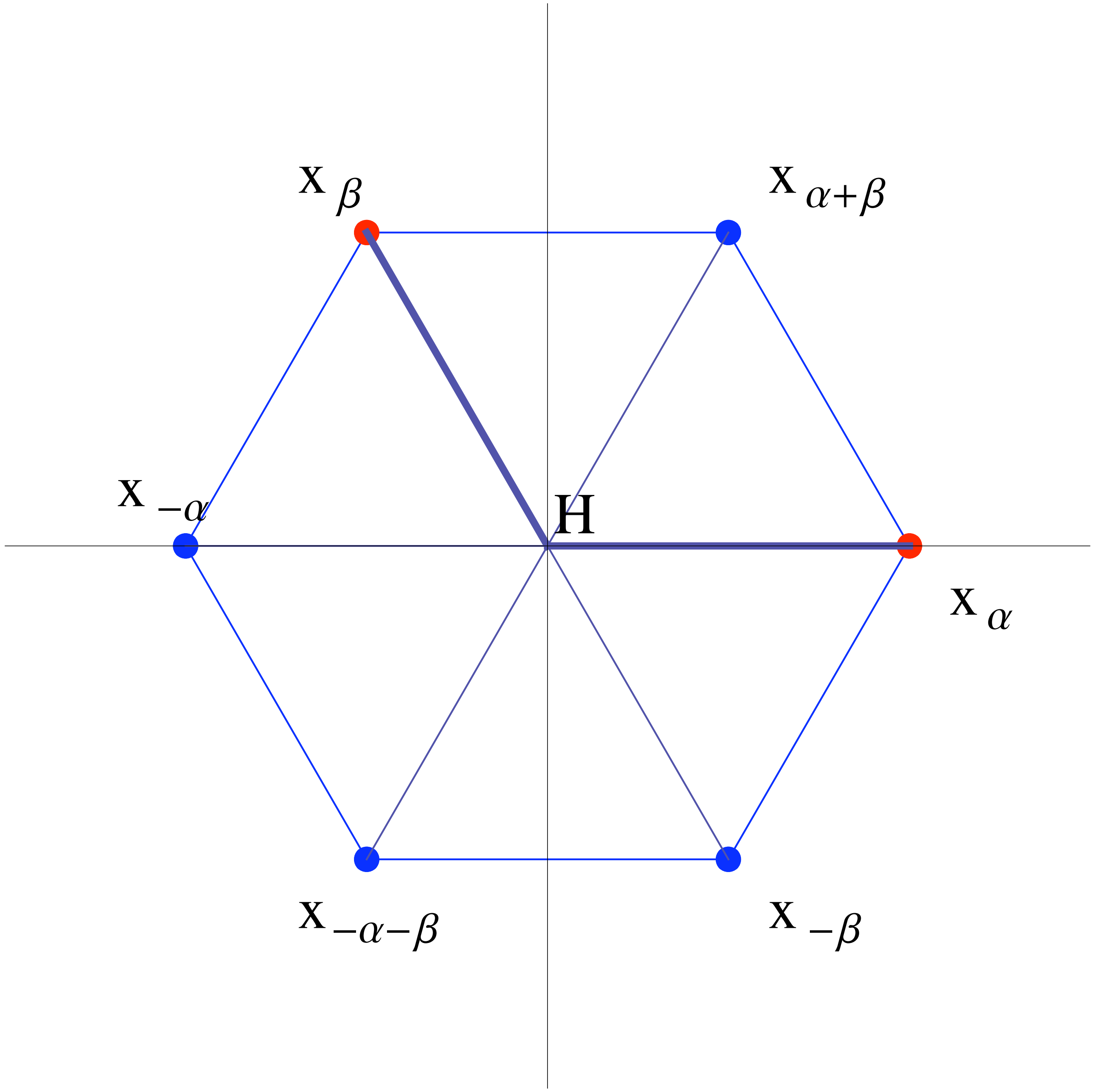
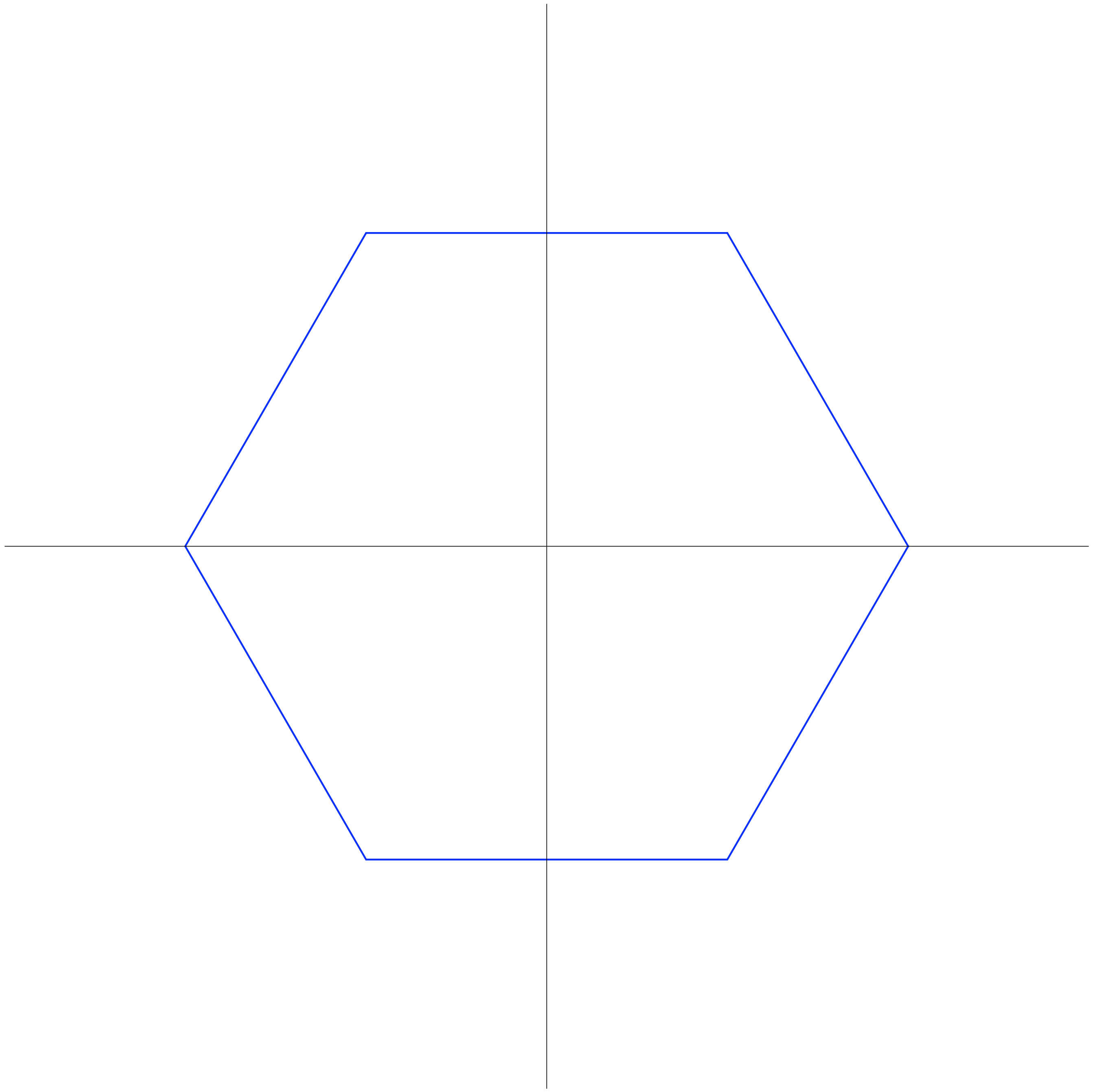
Chevalley Bases



Many Lie algebras have a *Chevalley basis*!

Root Systems

- **A hexagon**



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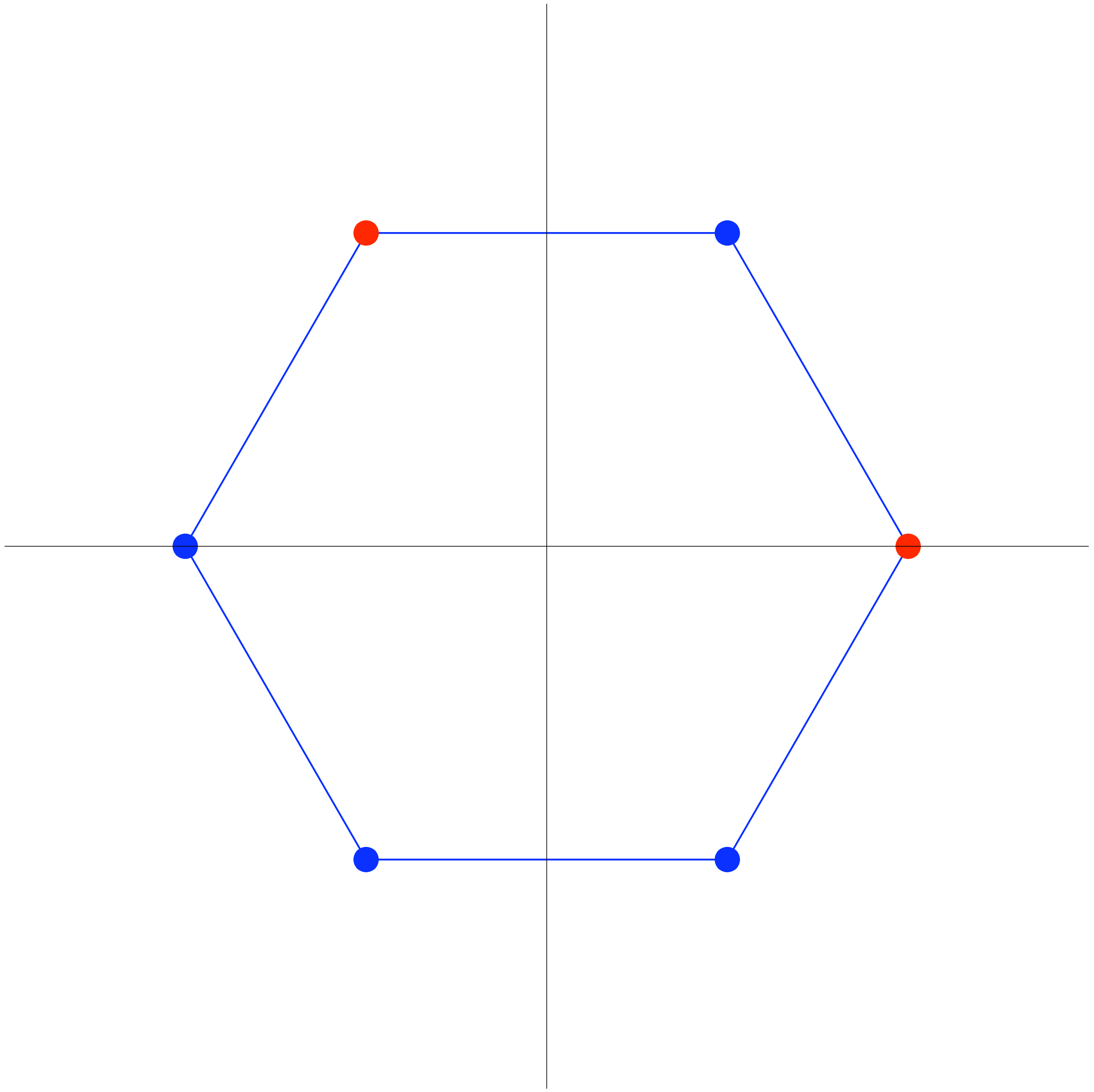

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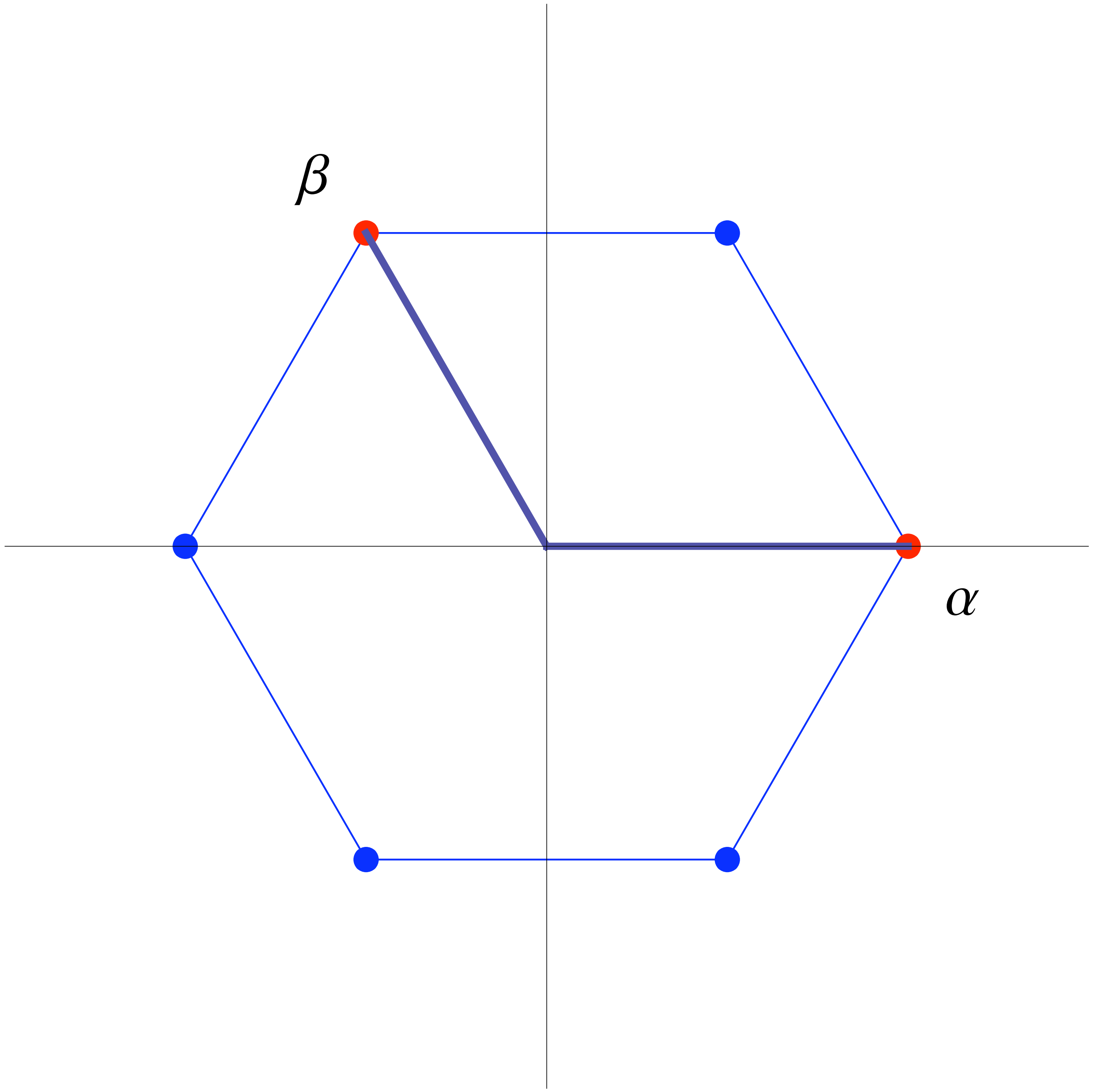
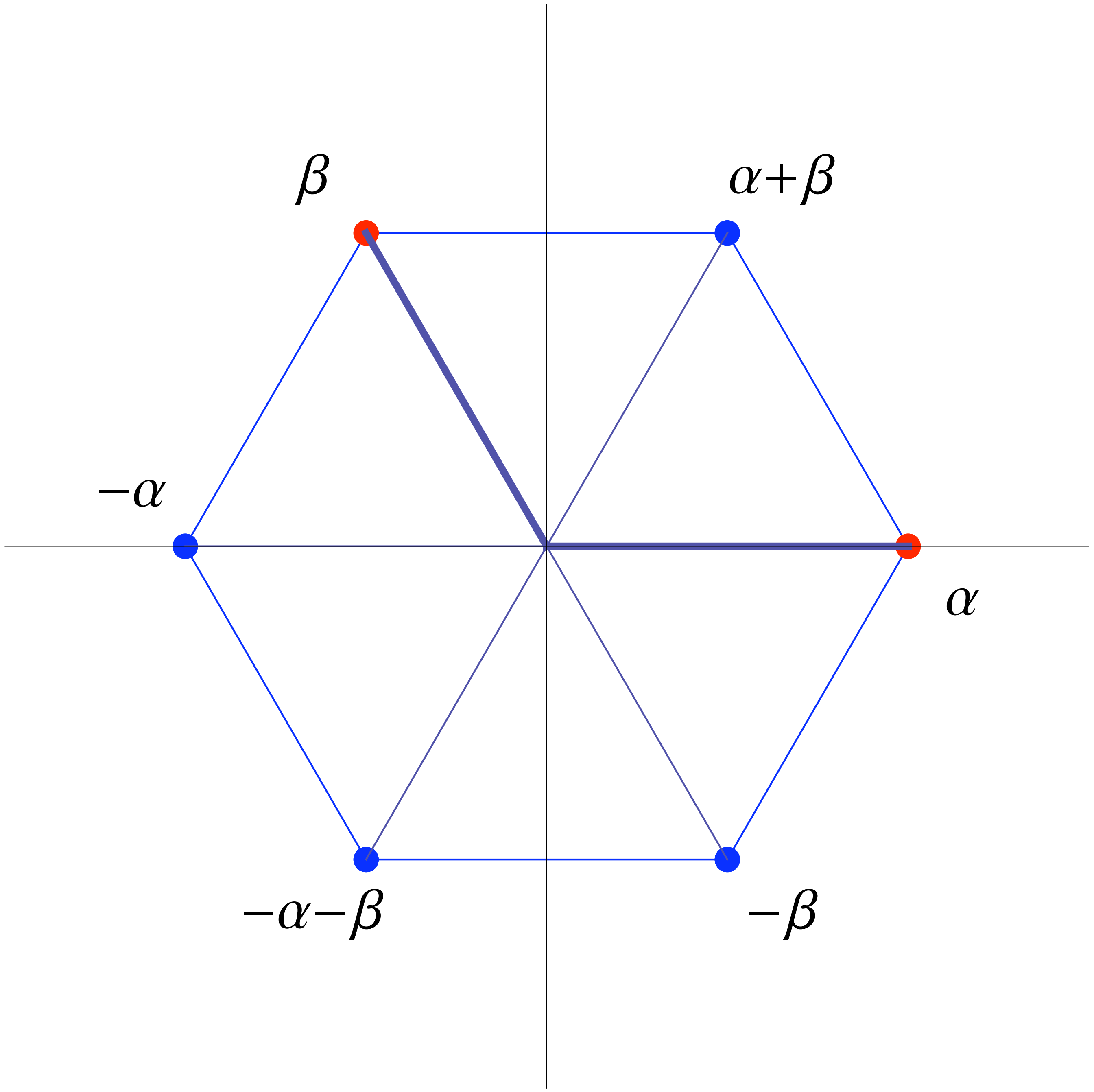


Root Systems

- A hexagon
- A root system of type A_2



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Definition (Root Datum)

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- put in duality by $\langle \cdot, \cdot \rangle$,
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One **Root System** $\longrightarrow$ $\left\{ \begin{array}{c} \textit{Several}\ \textbf{Root Data:} \\ \textbf{"adjoint"} \\ \vdots \\ \textbf{"simply connected"} \end{array} \right.$

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- **"adjoint"**
- ⋮
- **"simply connected"**

Irreducible Root Data: $\mathrm{A}_n^{\cdot}, \mathrm{B}_n^{\cdot}, \mathrm{C}_n^{\cdot}, \mathrm{D}_n^{\cdot}, \mathrm{E}_6^{\cdot}, \mathrm{E}_7^{\cdot}, \mathrm{E}_8^{\cdot}, \mathrm{F}_4^{\cdot}, \mathrm{G}_2^{\cdot}$.

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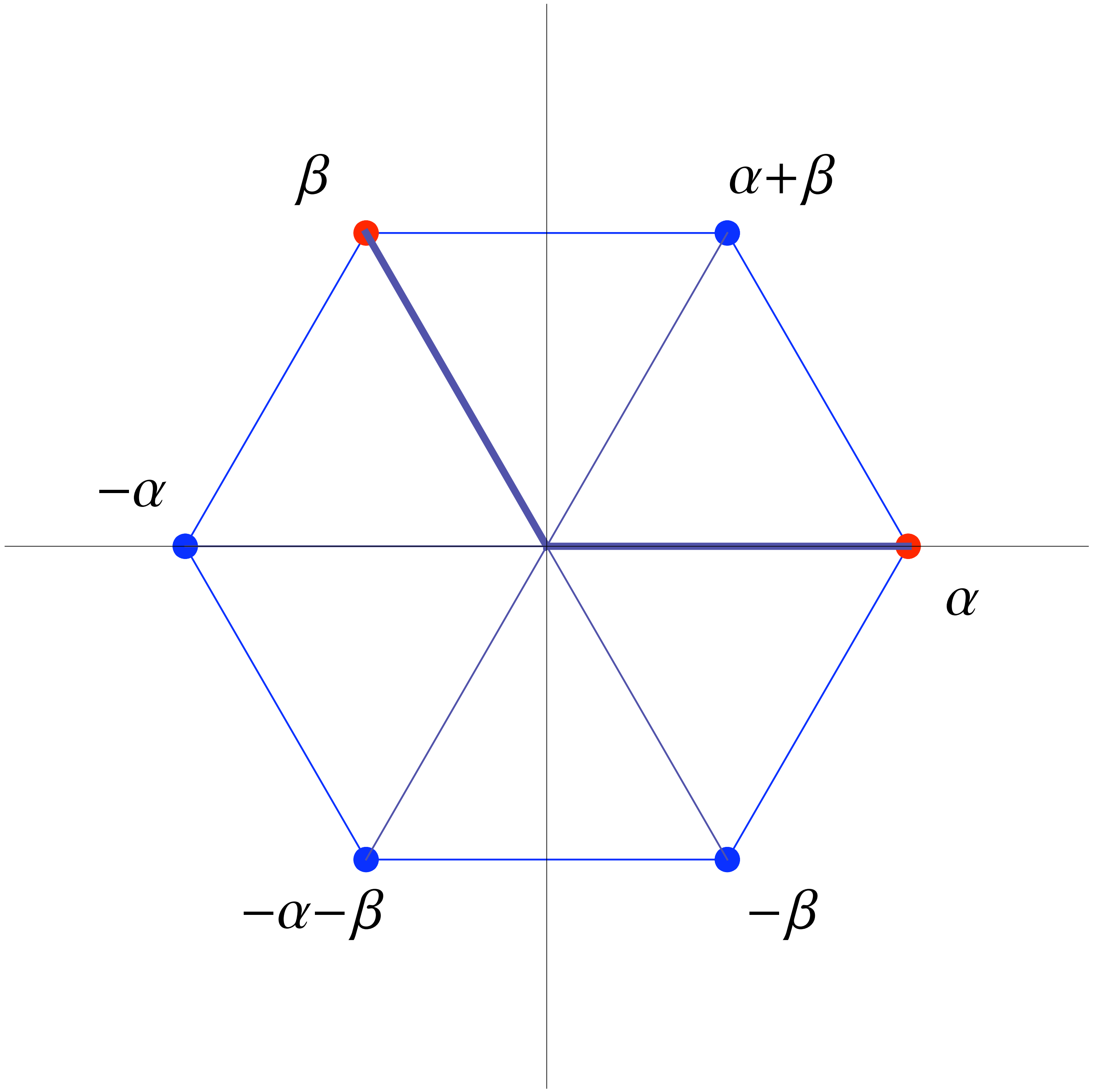
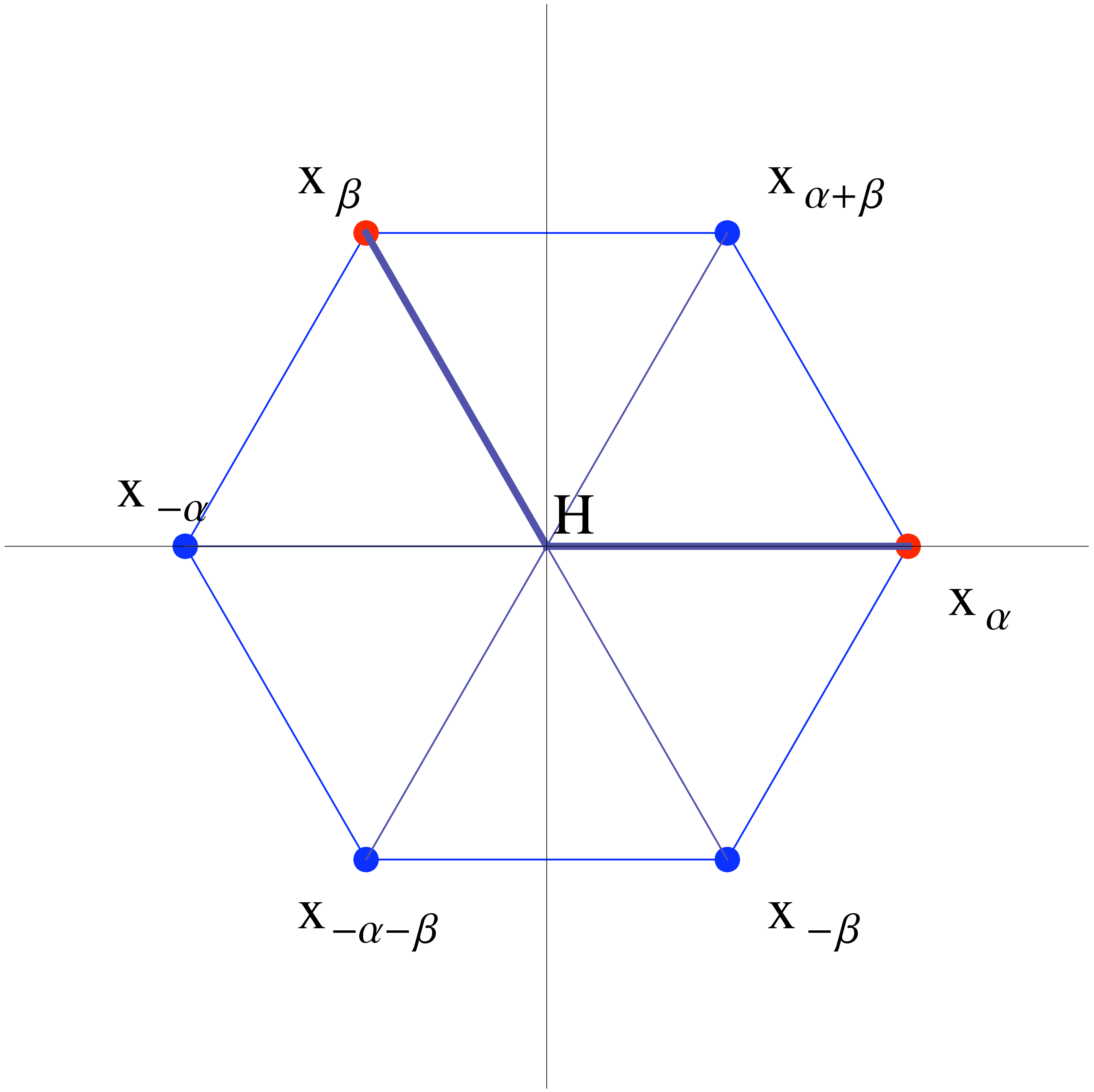


Root Data

- A hexagon
- A root system of type A_2
- A Lie algebra of type A_2



Chevalley Basis

Definition (Chevalley Basis)

Formal basis: $L = \bigoplus_{i=1,\ldots,n} \mathbb{F}h_i \oplus \bigoplus_{\alpha\in\Phi} \mathbb{F}x_\alpha$

Bilinear anti-symmetric multiplication satisfies ($i, j \in \{1, \ldots, n\}; \alpha, \beta \in \Phi$) :

$$\begin{aligned}
[h_i, h_j] &= 0, \\
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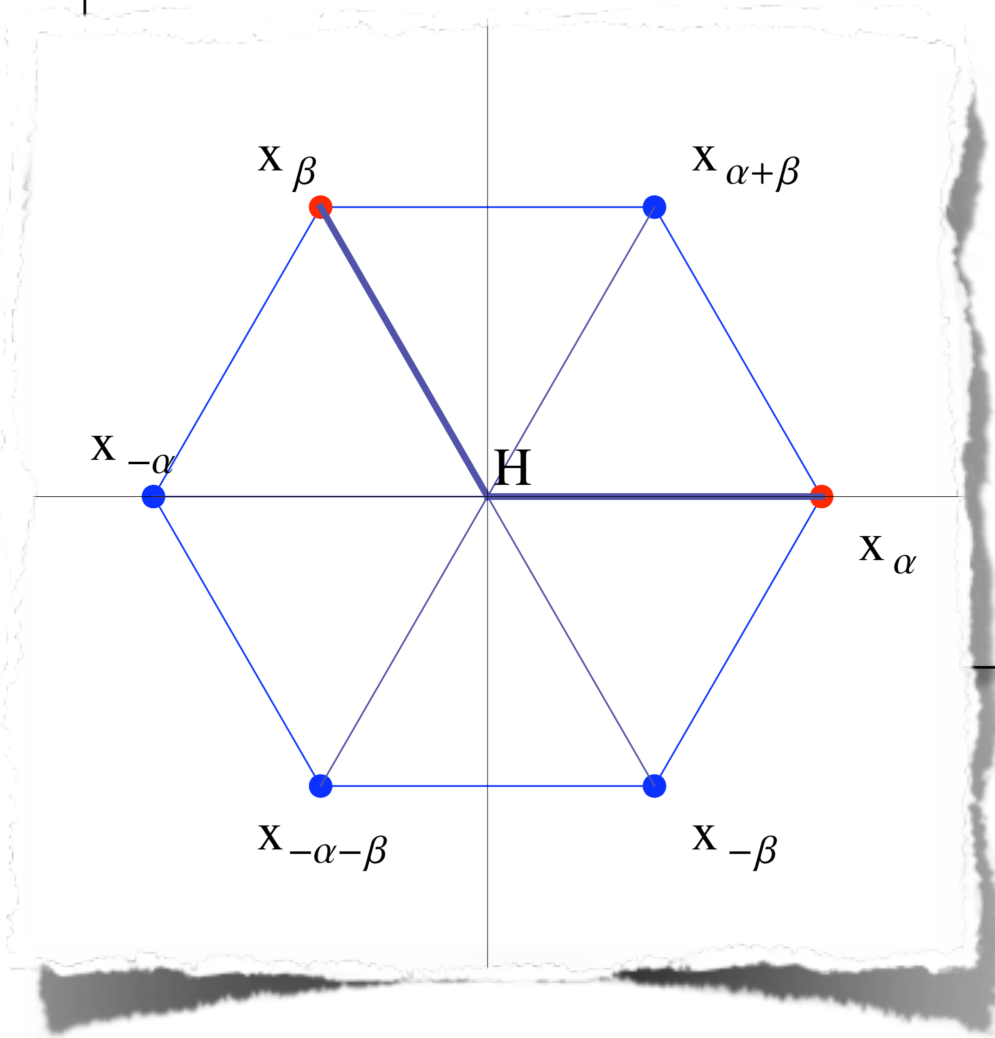

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Why Chevalley bases?

- Because transformation between two Chevalley bases is an automorphism of L,
- So we can test isomorphism between two Lie algebras (and find isomorphisms!) by computing Chevalley bases.

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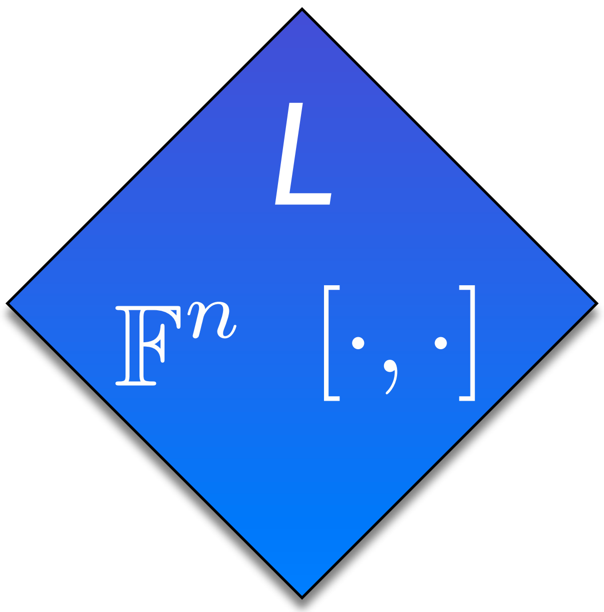
Why?



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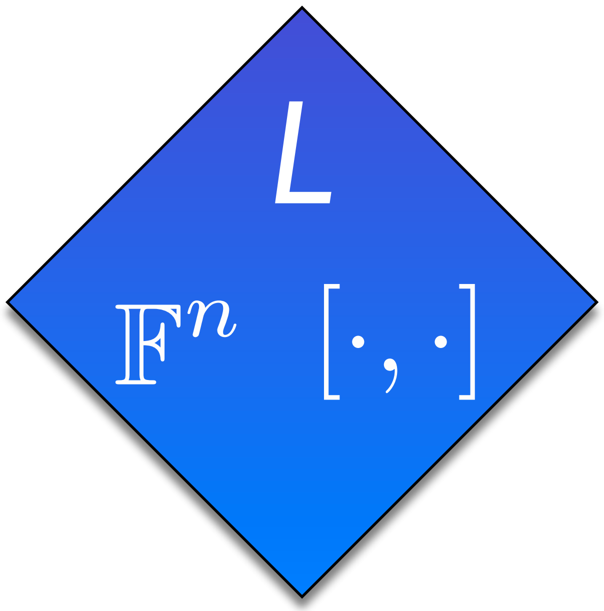
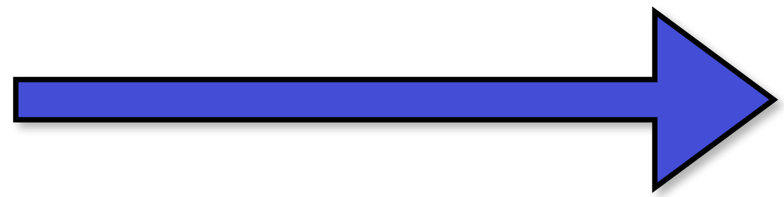
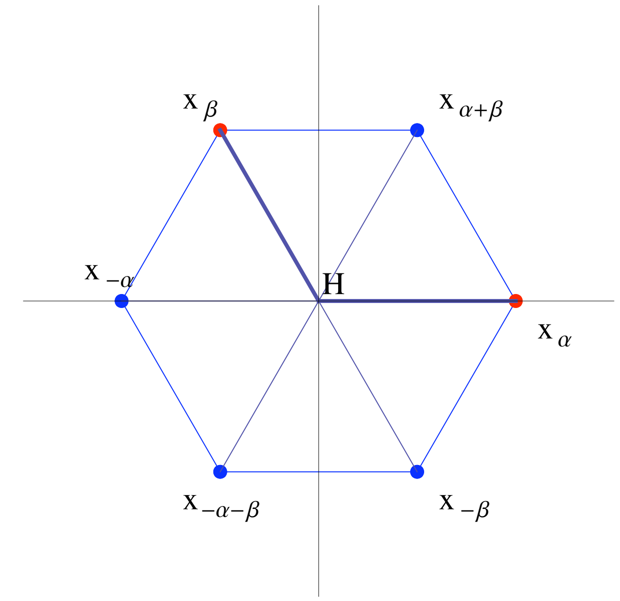

L
$\mathbb{F}^n$
$[\cdot,\cdot]$




x_β
$x_{\alpha+\beta}$
$x_{-\alpha}$
H
x_α
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$x_{-\beta}$

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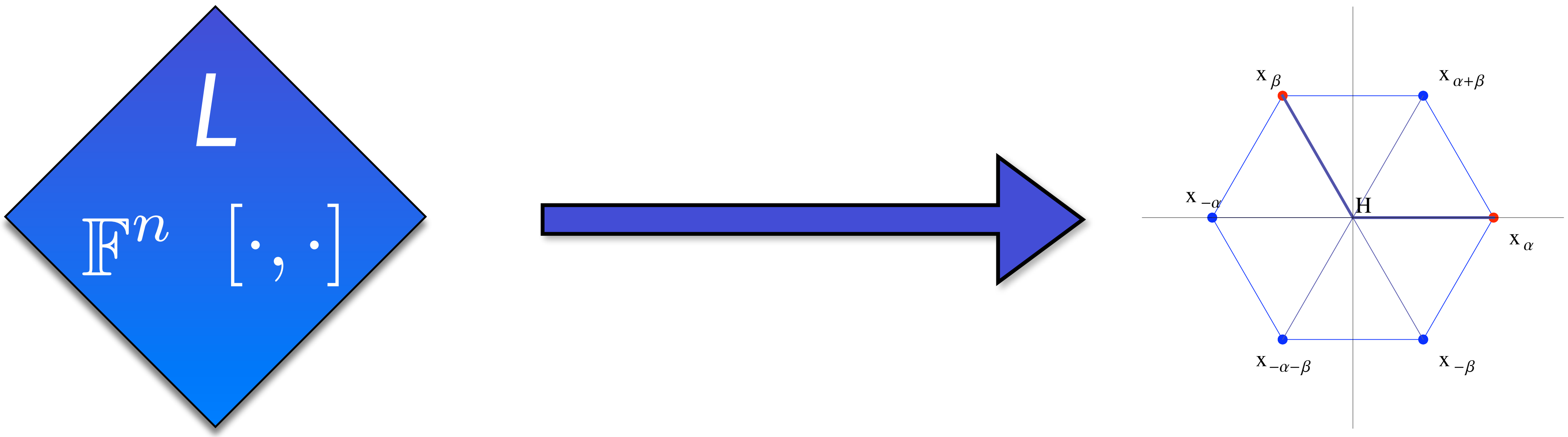
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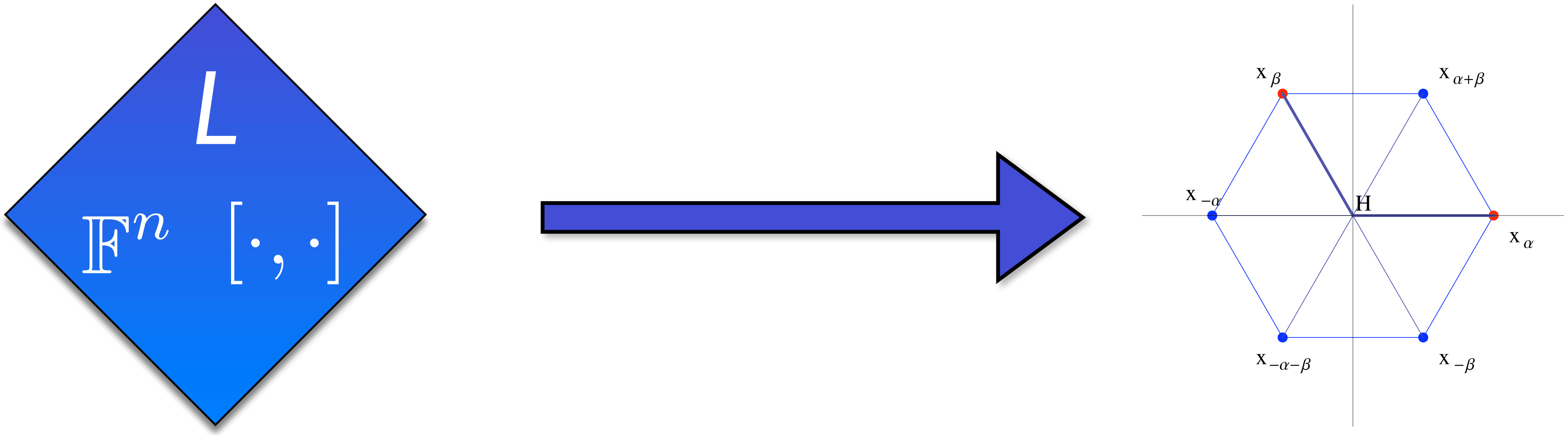

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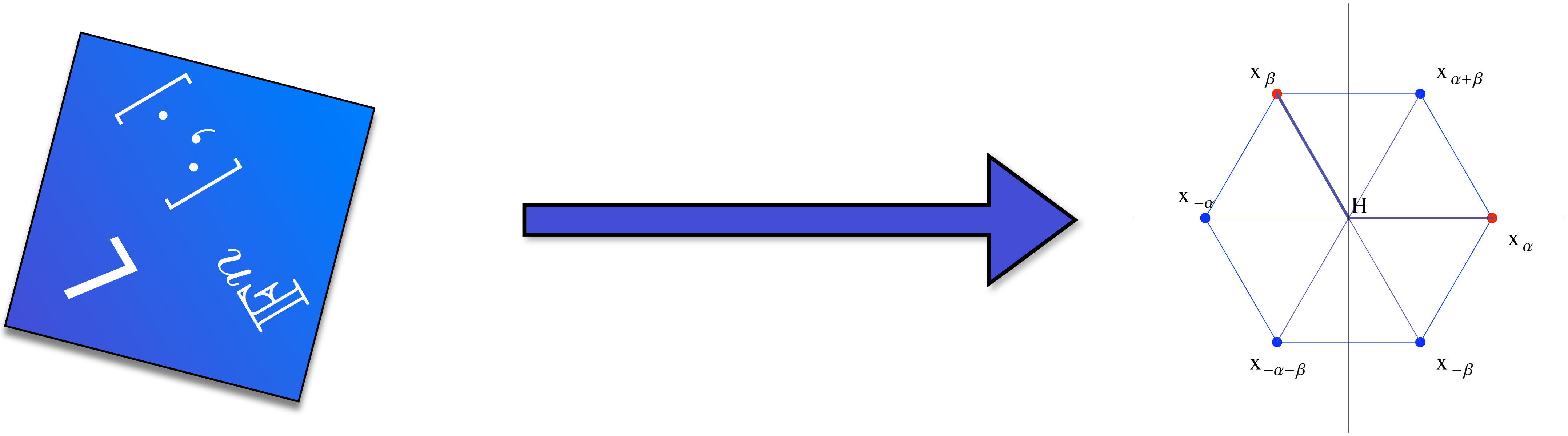

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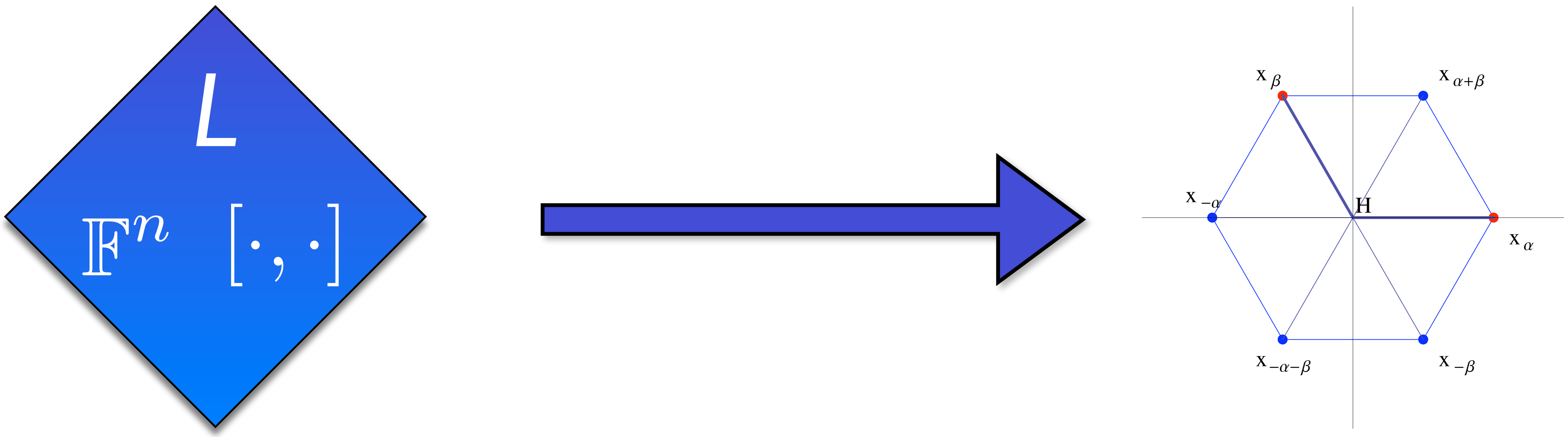
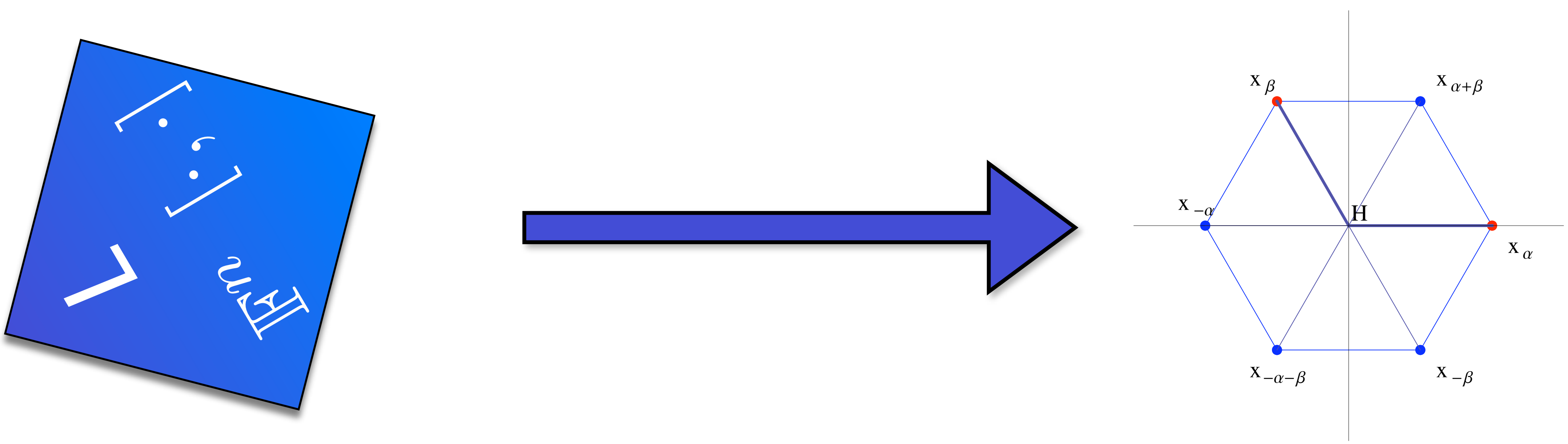


equal



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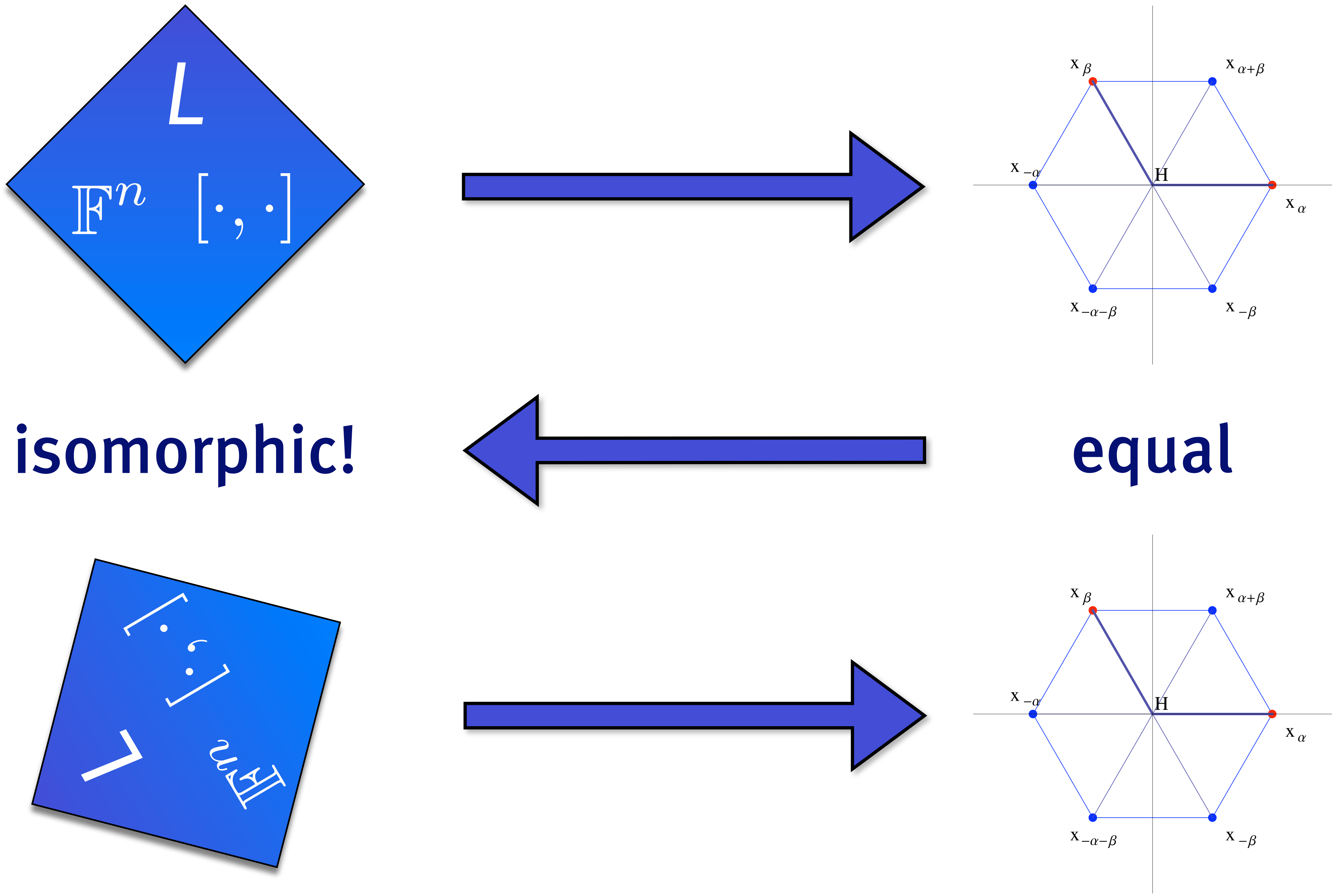

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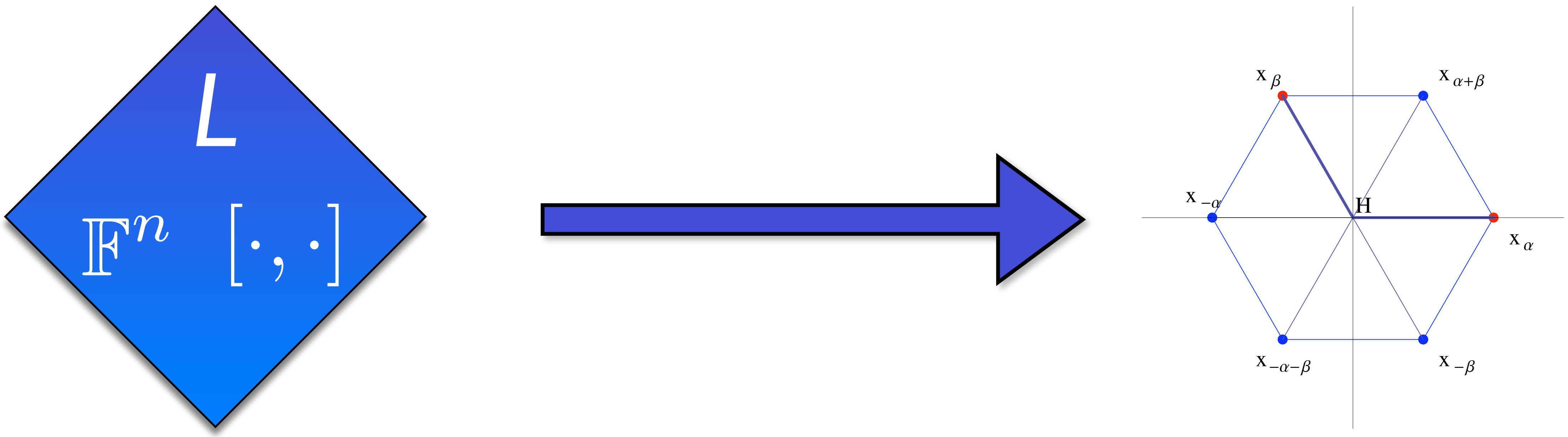

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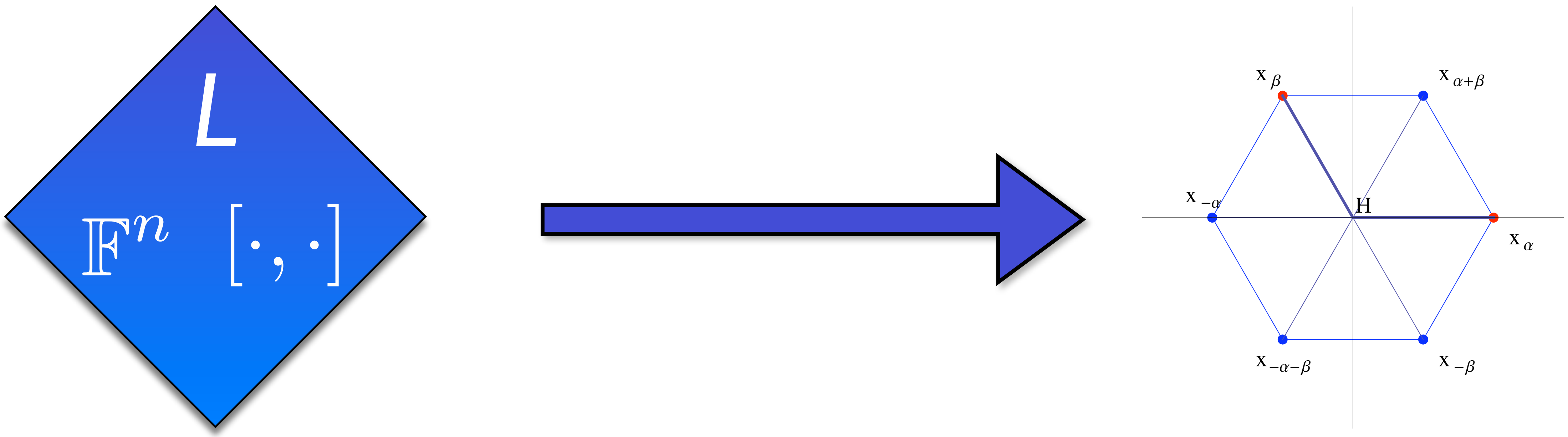
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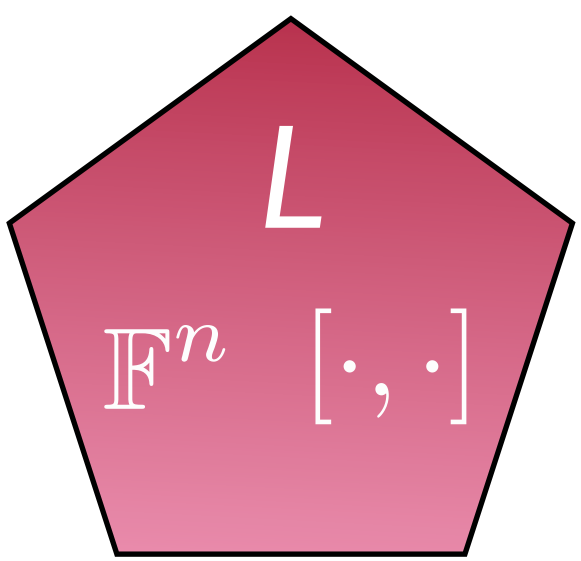
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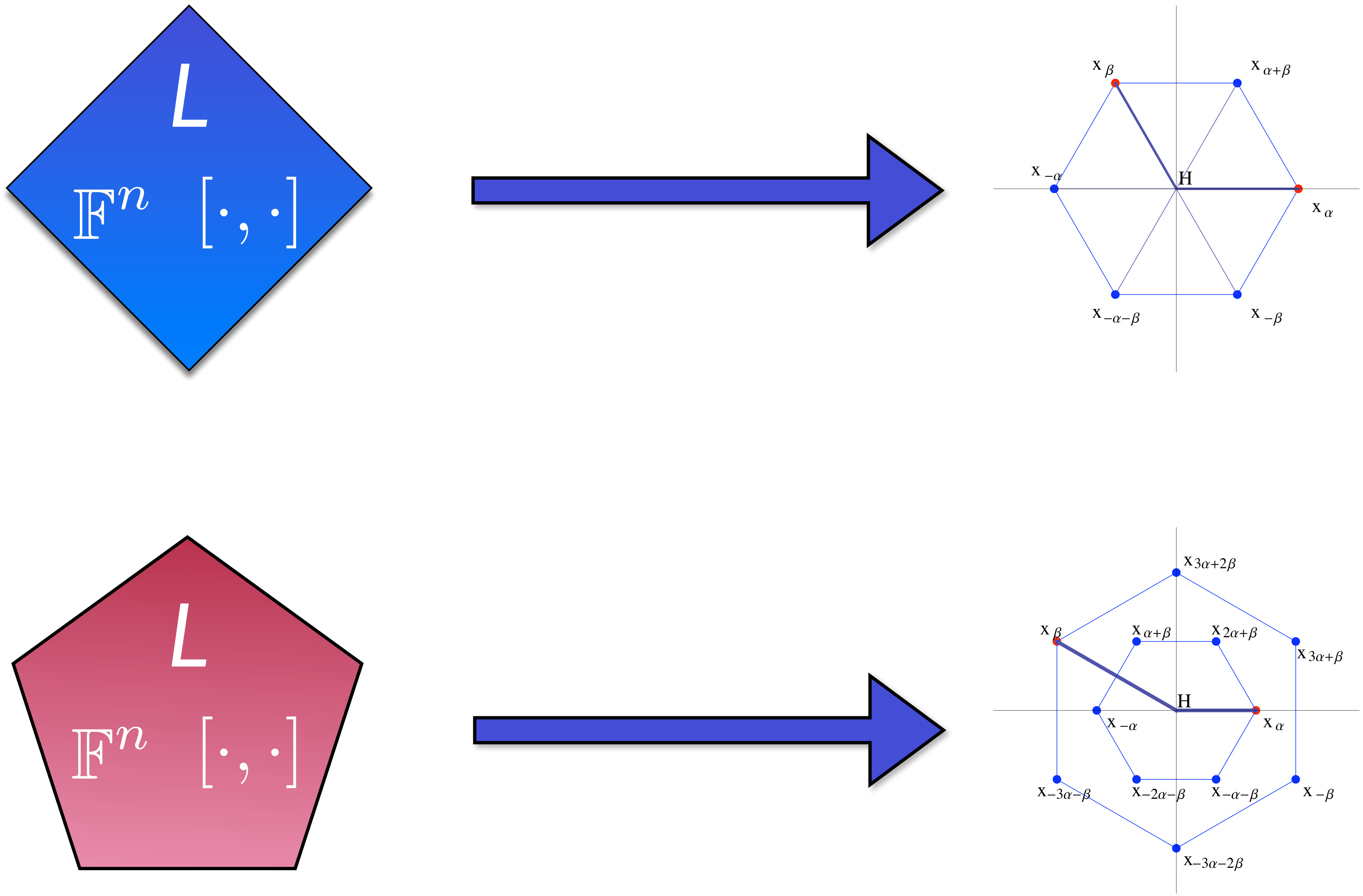

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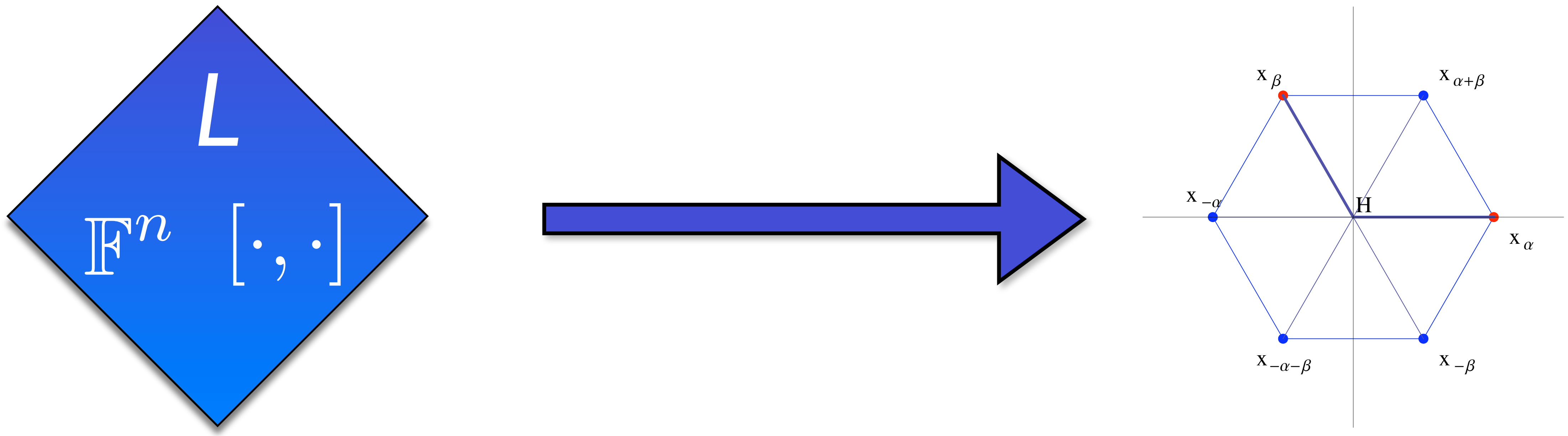
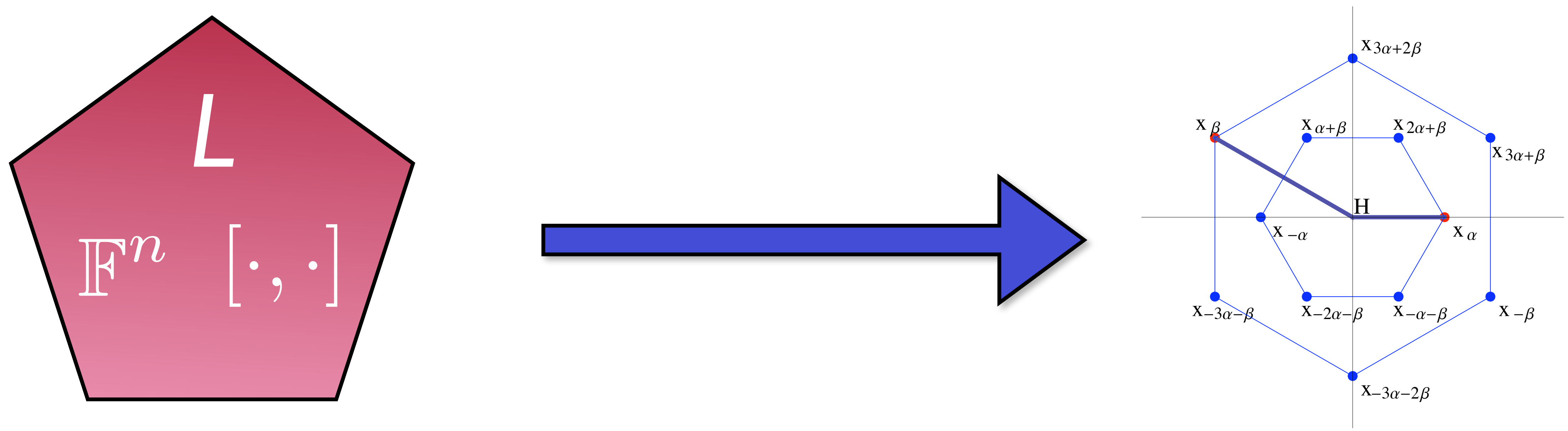


not equal



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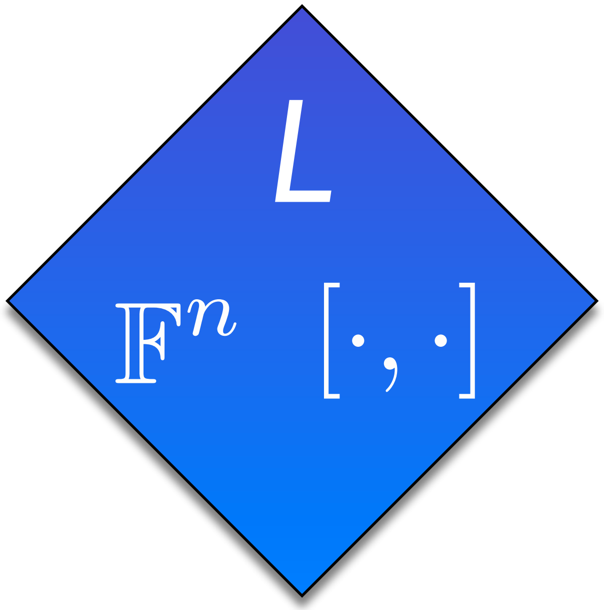
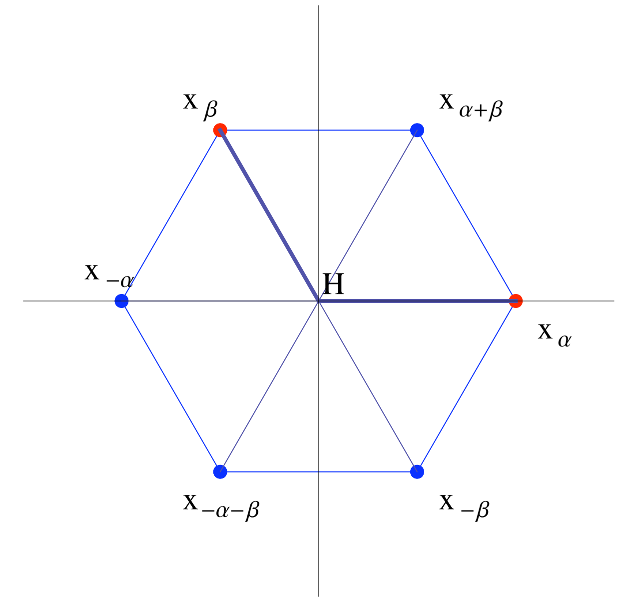
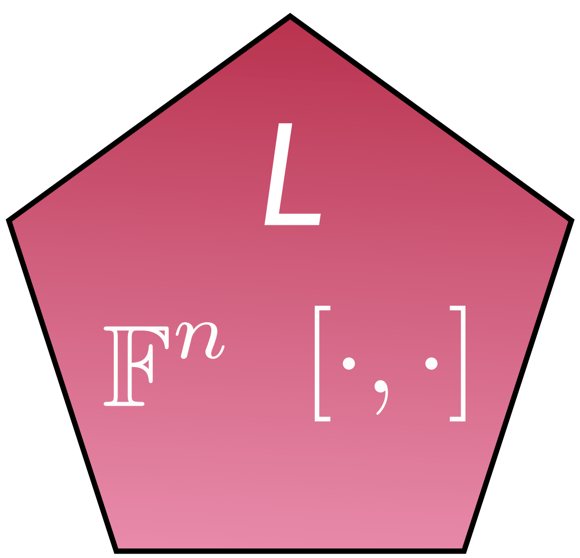
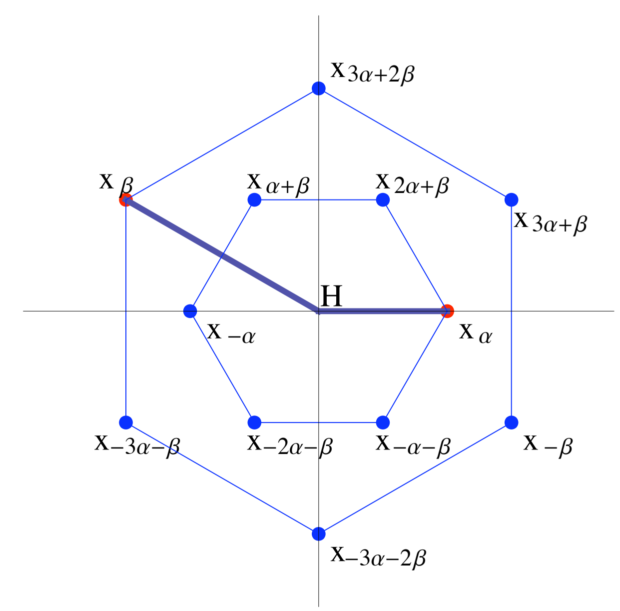




non-isomorphic! ← **not equal**





Outline

- What is a Lie algebra?
- What is a Chevalley basis?
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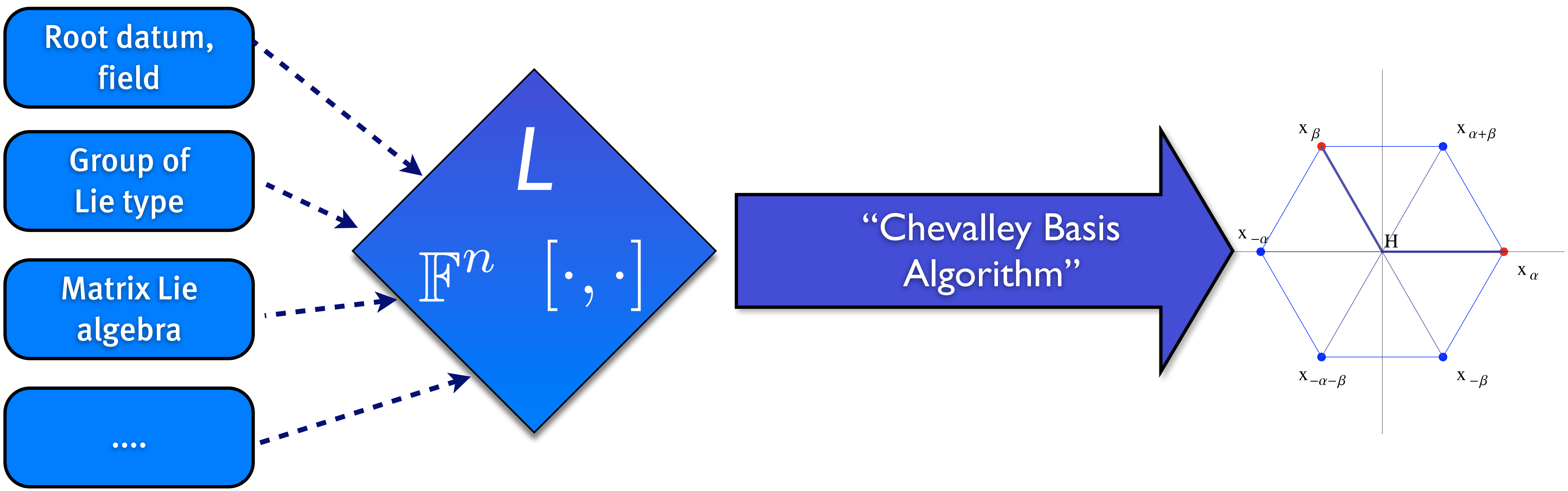
The Mission

- Given a Lie algebra (on a computer),
- Want to know which Lie algebra it is,
- So want to compute a Chevalley basis for it.



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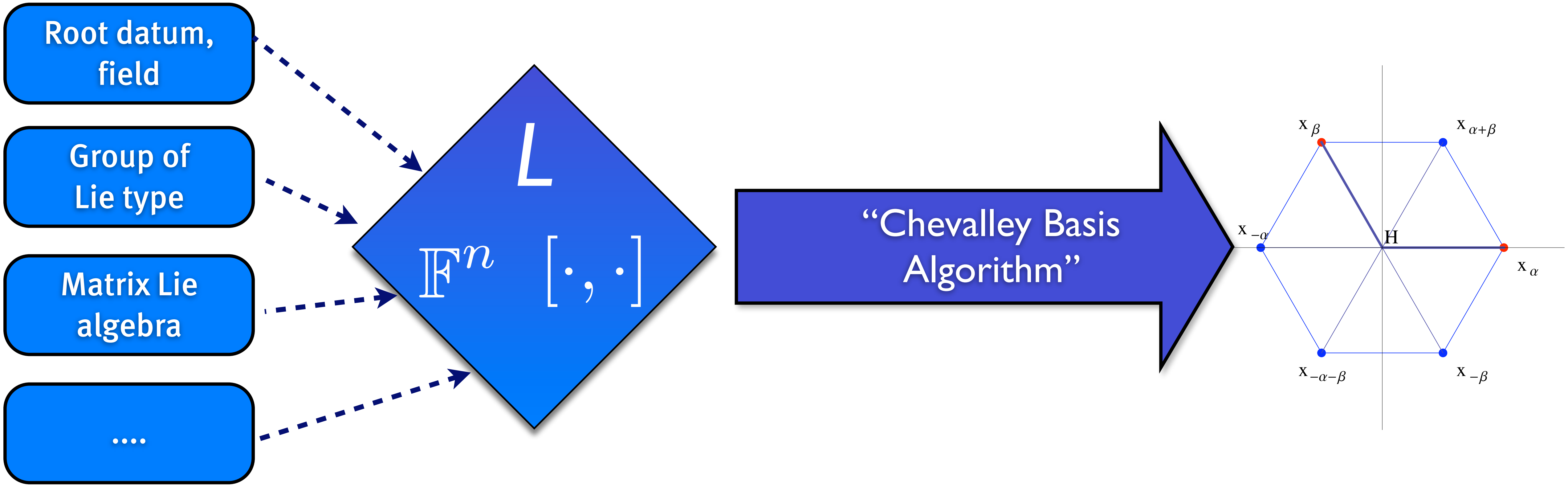
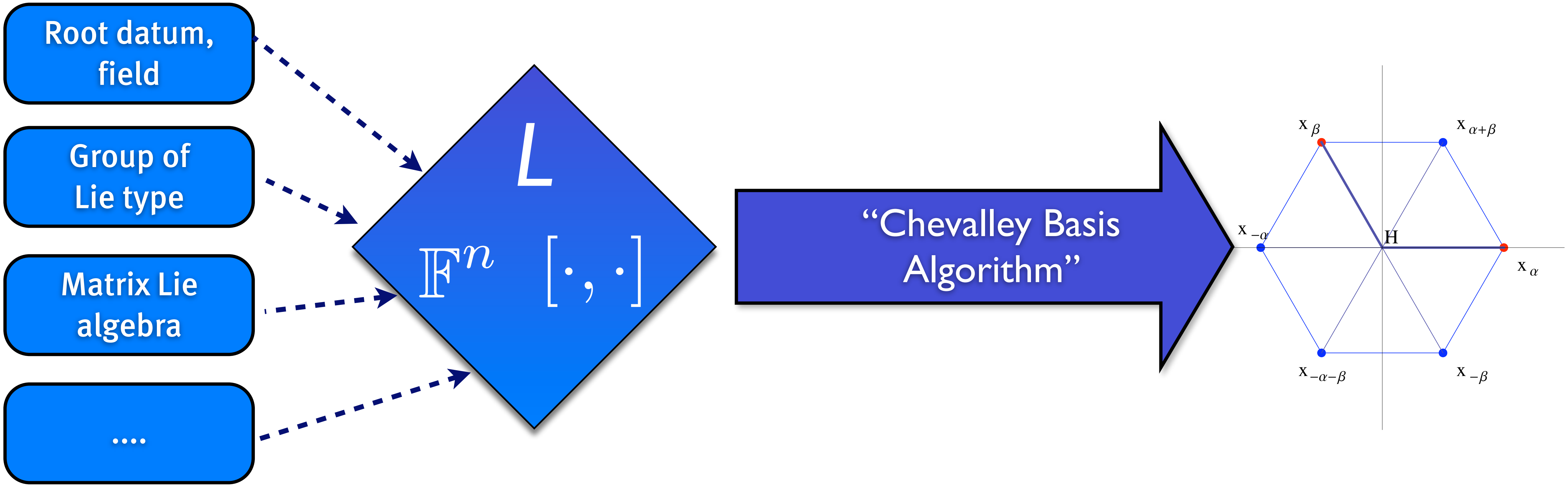


- Assume *splitting Cartan subalgebra* **H is given (Cohen/Murray, indep. Ryba);**
- **Assume root datum R is given**

The Mission



- **Char. 0, p ≥ 5: De Graaf, Murray; implemented in GAP, MAGMA**

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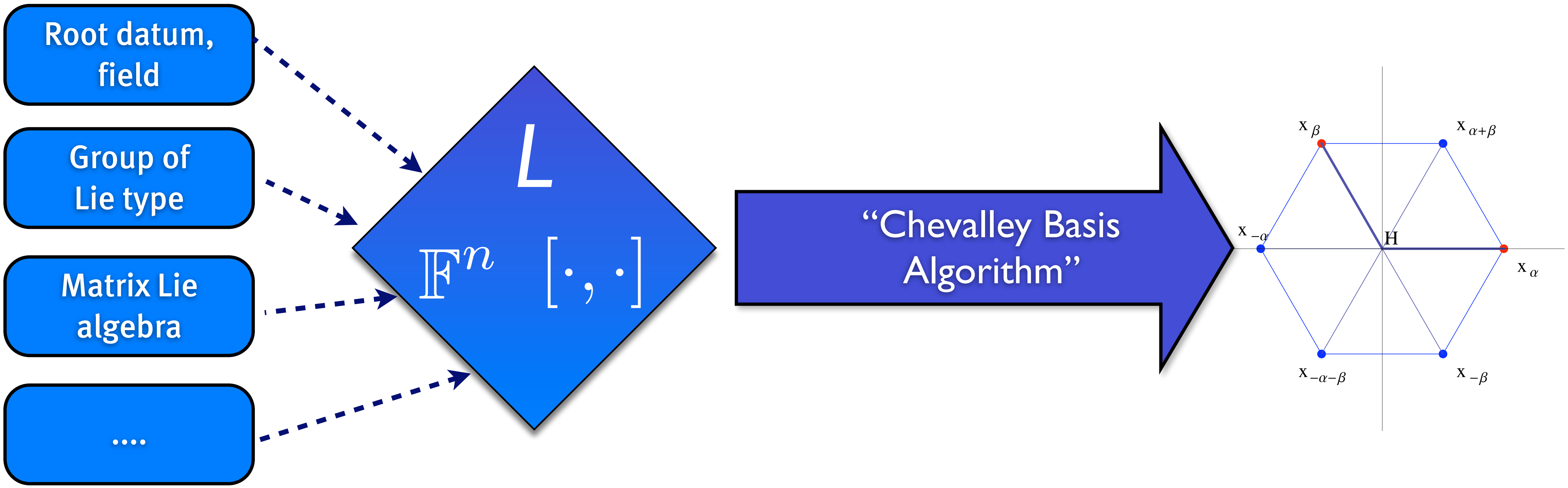
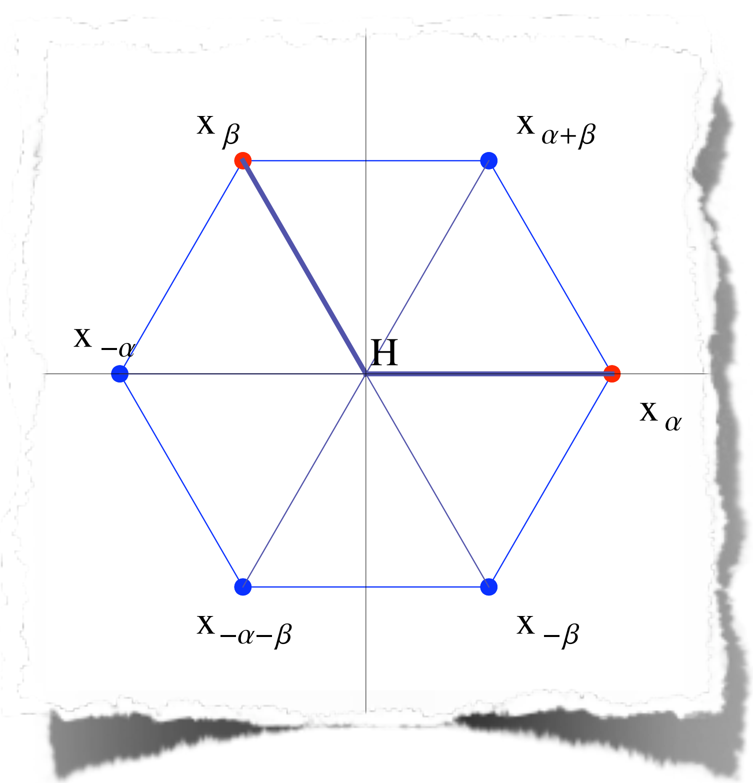


- Char. 0, p ≥ 5: De Graaf, Murray; implemented in GAP, MAGMA
- Char. 2,3: R., 2009, Implemented in MAGMA

The Problems

Normally:

- Diagonalise L using action of H on L (gives set of x_α),
- Use Cartan integers $\langle \alpha, \beta \rangle$ to "identify" the x_α ,
- Solve easy linear equations.



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and the Jacobi identity.

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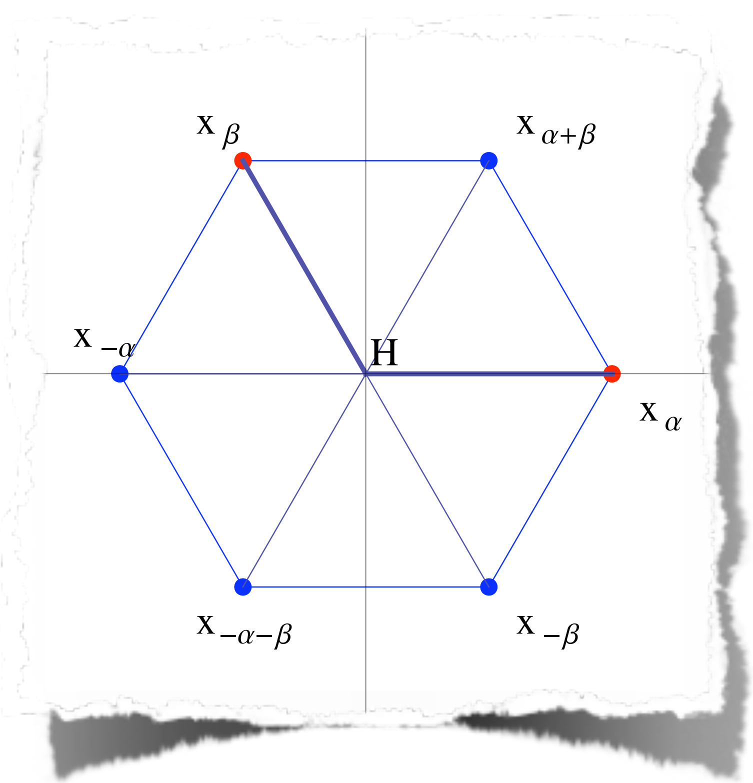

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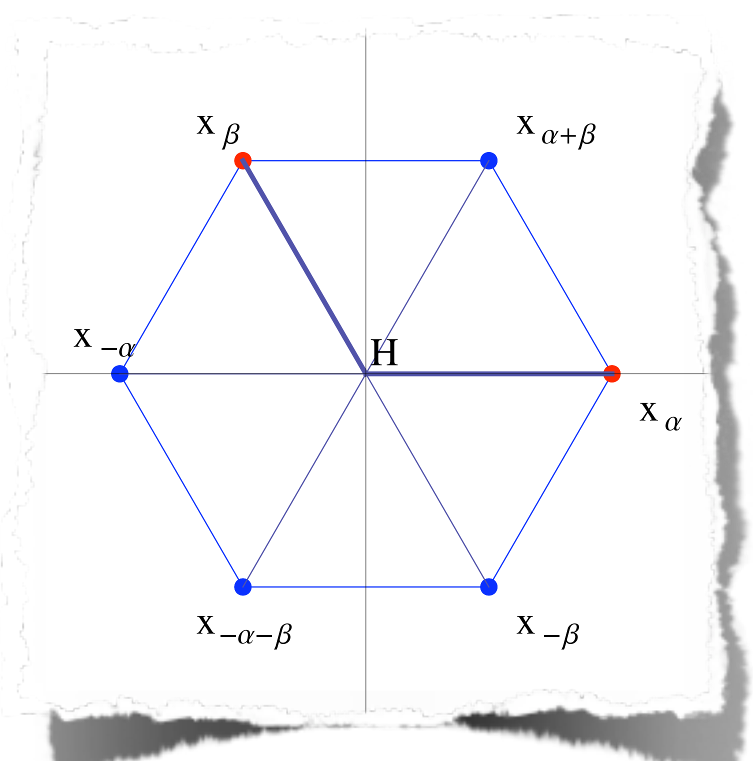

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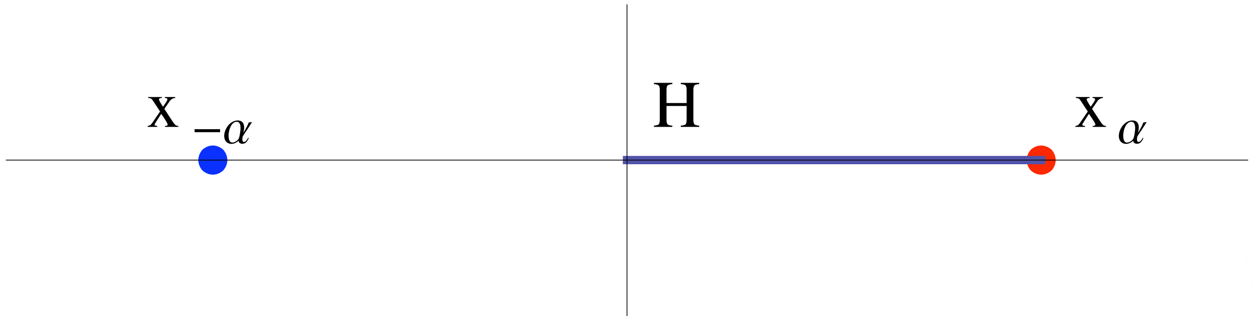
Diagonalising (A_1, char. 2)



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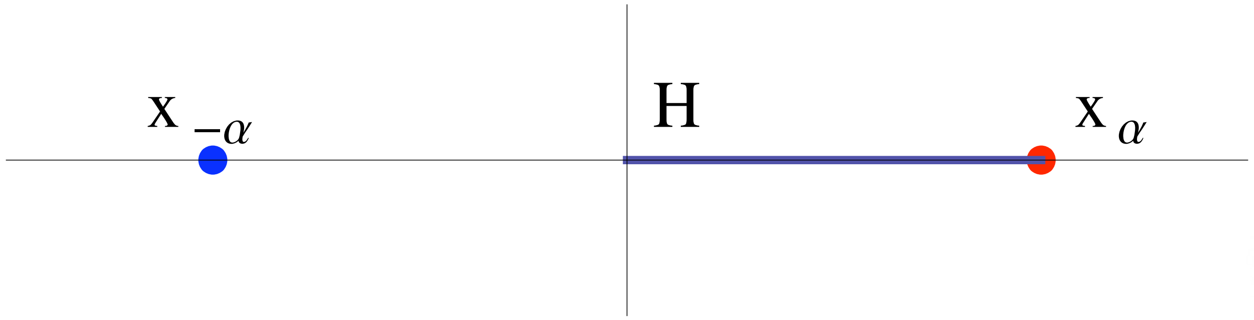

$$A_1^{Ad} : X = Y = \mathbb{Z}$$

$$\Phi = \{\alpha = 1, -\alpha = -1\},$$

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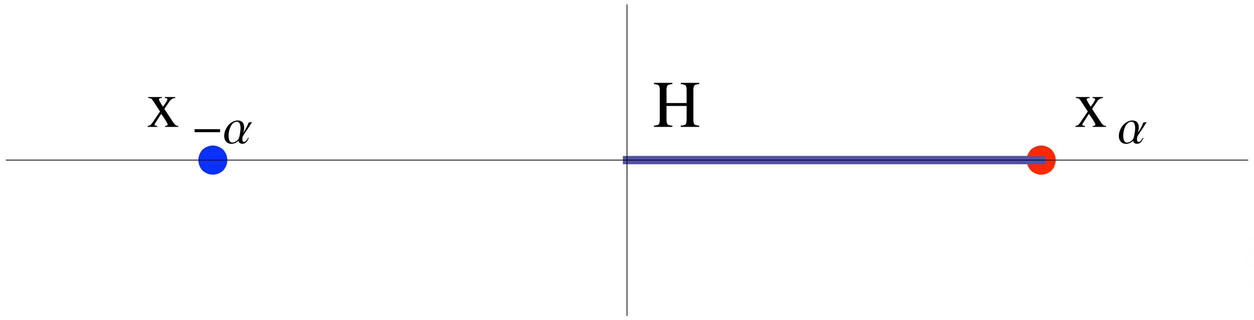
Diagonalising (A₁, char. 2)



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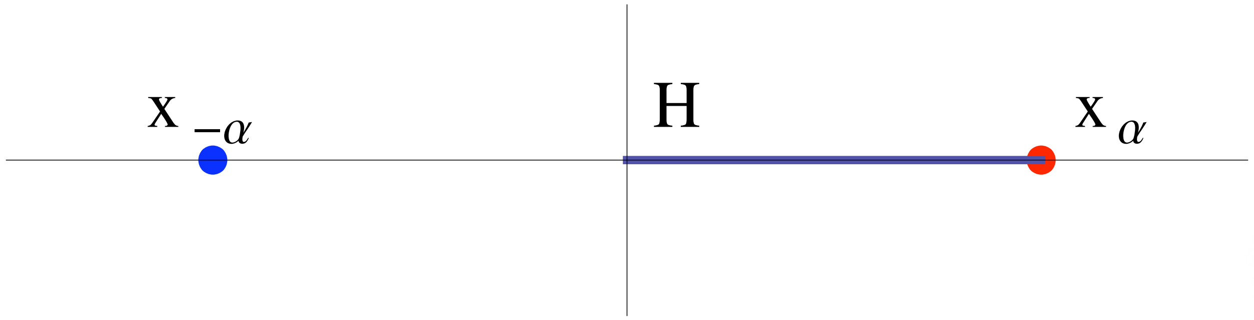
[h_i, h_j] &= 0,\\
[x_\alpha, h_i] &= \langle \alpha, f_i\rangle x_\alpha,\\
[x_{-\alpha}, x_\alpha] &= \textstyle\sum_{i=1}^{n} \langle e_i, \alpha^\vee\rangle h_i,\\
[x_\alpha, x_\beta] &= \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

and the Jacobi identity.

Diagonalising (A₁, char. 2)



$$\mathrm{A}_1^{\mathrm{Ad}} : X = Y = \mathbb{Z}$$

$$\Phi = \{\alpha = 1, -\alpha = -1\},$$

$$\Phi^\vee = \{\alpha^\vee = 2, -\alpha^\vee = -2\},$$

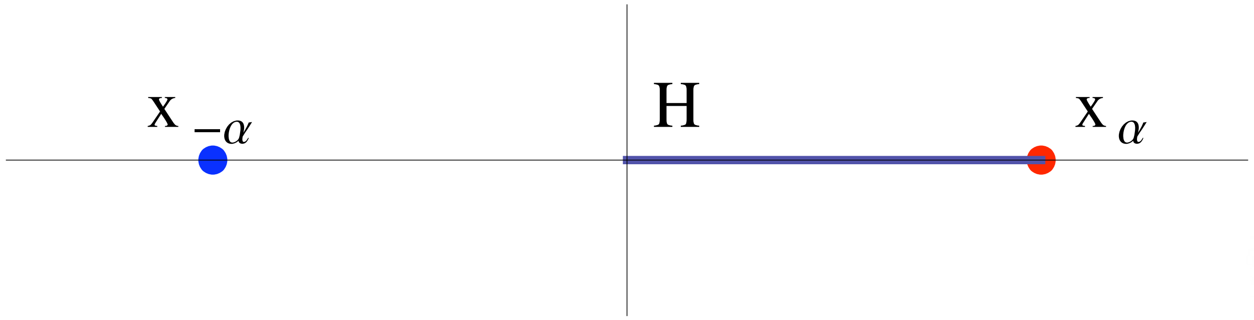
$$L = \mathbb{F}h \oplus \mathbb{F}x_\alpha \oplus \mathbb{F}x_{-\alpha}$$

	x_α	$x_{-\alpha}$	h
x_α	0	$\langle e_1, \alpha^\vee \rangle h$	$\langle \alpha, f_1 \rangle x_\alpha$
$x_{-\alpha}$		0	$\langle -\alpha, f_1 \rangle x_{-\alpha}$
h			0

$$\begin{aligned} [h_i, h_j] &= 0, \\ [x_\alpha, h_i] &= \langle \alpha, f_i \rangle x_\alpha, \\ [x_{-\alpha}, x_\alpha] &= \textstyle\sum_{i=1}^n \langle e_i, \alpha^\vee \rangle h_i, \\ [x_\alpha, x_\beta] &= \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and the Jacobi identity.

Diagonalising (A_1, char. 2)



$$A_1^{\mathrm{Ad}} : X = Y = \mathbb{Z}$$

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$$
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[x_{-\alpha}, x_\alpha] &= \textstyle\sum_{i=1}^n \langle e_i, \alpha^\vee \rangle h_i, \\
[x_\alpha, x_\beta] &= \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
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h			0

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

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Diagonalising (A_1, char. 2)

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x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
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Diagonalising (A$_1$, char. 2)

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x_α	0	$-2h$	x_α
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Basis transformation....

$$x = x_\alpha - x_{-\alpha}$$

$$y = 2x_\alpha + x_{-\alpha}$$

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Basis transformation....

$$\begin{aligned} x &= x_\alpha - x_{-\alpha} \\ y &= 2x_\alpha + x_{-\alpha} \end{aligned}$$

	x	y	h
x	0	$-6h$	$-\frac{1}{3}x + \frac{2}{3}y$
y	$6h$	0	$\frac{4}{3}x + \frac{1}{3}y$
h	$\frac{1}{3}x - \frac{2}{3}y$	$-\frac{4}{3}x - \frac{1}{3}y$	0

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
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$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

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Algorithm:

- **Diagonalize L wrt H**
- **Find 1-dim eigenspaces:**

  $$S_1, S_{-1}, S_0$$
- **Take**

  $$x + y \in S_1$$
  $$x - \tfrac{1}{2}y \in S_{-1}$$
  $$h \in S_0$$
- **Done!**

Diagonalising (A_1, char. 2)

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

Basis transformation....

$$x = x_\alpha - x_{-\alpha}$$
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Algorithm:

- **Diagonalize L wrt H**
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$$x + y \in S_1$$
$$x - \frac{1}{2}y \in S_{-1}$$
$$h \in S_0$$

- **Done!**
- **But if char. is 2...**

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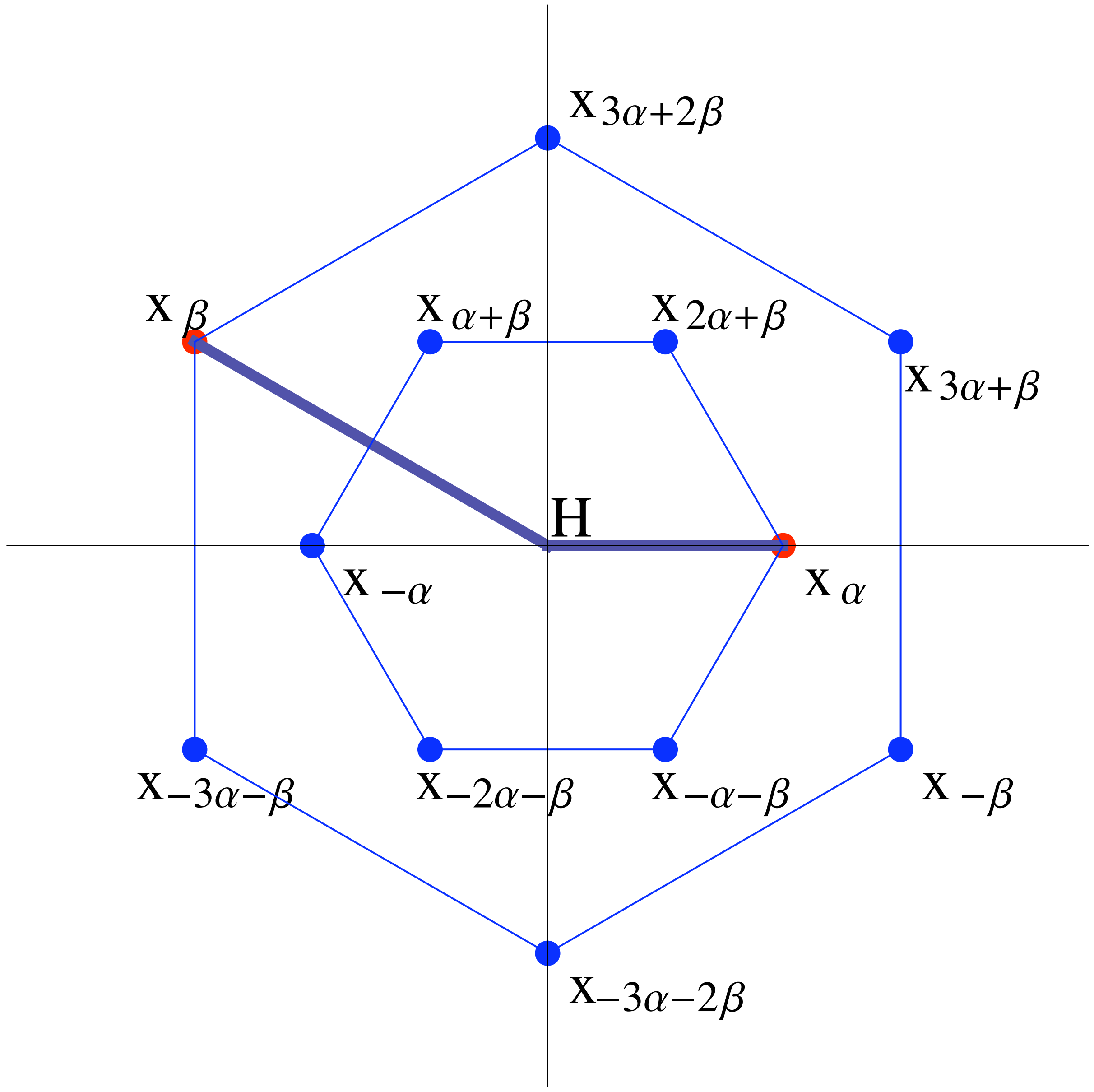
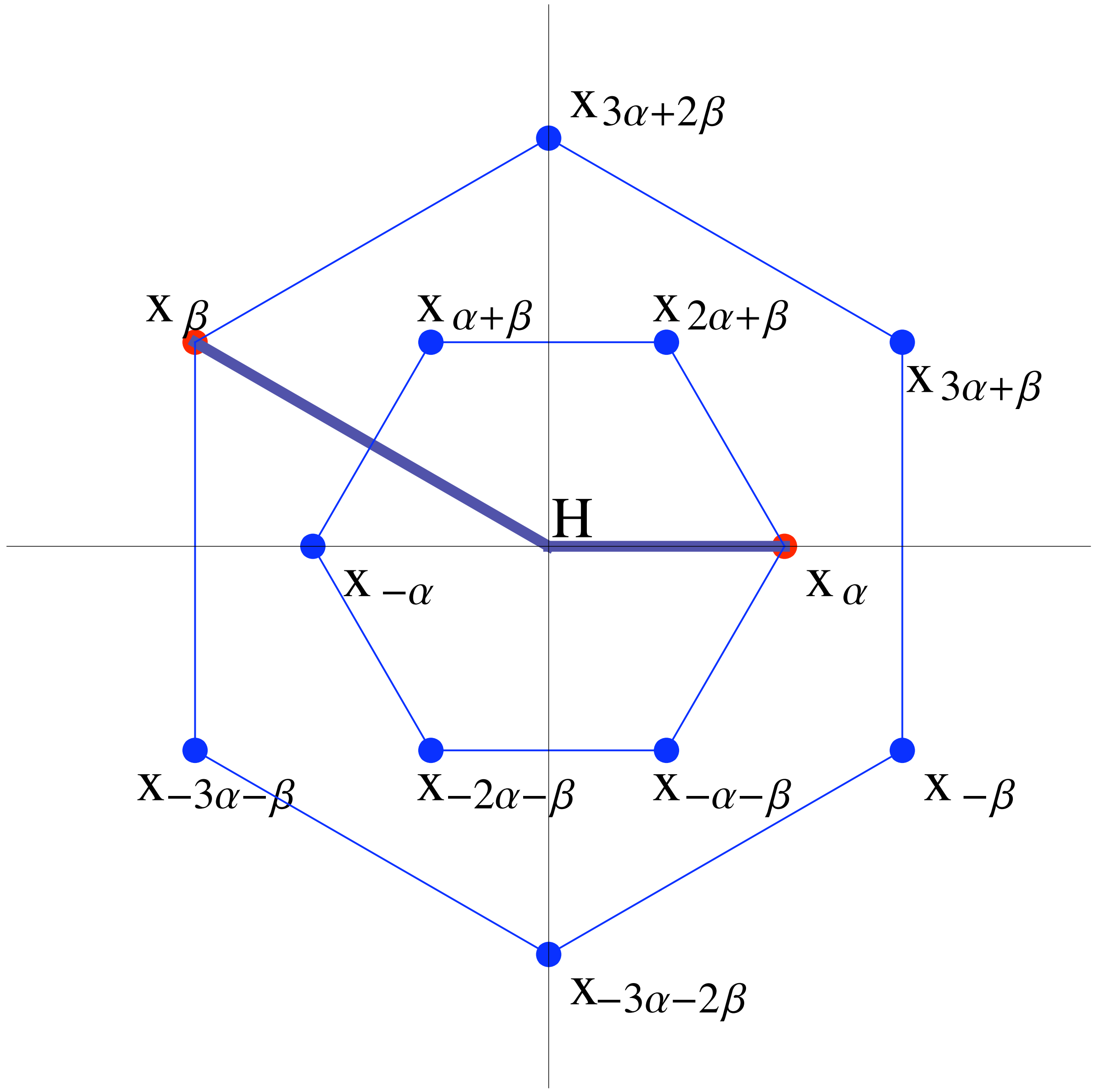
Diagonalising (G_2, char. 3)

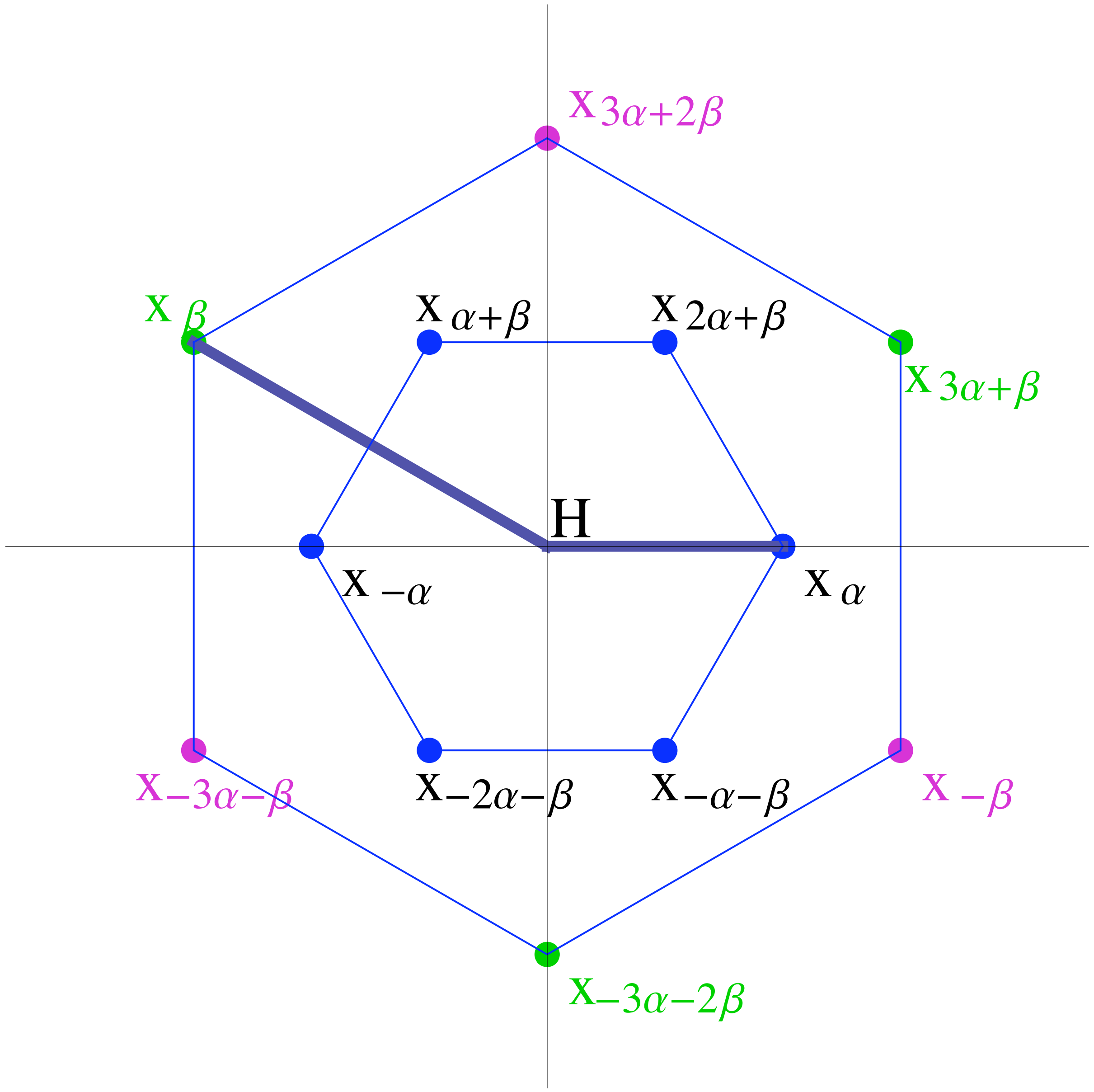


char. not 3

Diagonalising (G_2, char. 3)



char. not 3



char. 3

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Diagonalising (overview)

$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$\mathrm{A_2}^{\mathrm{sc}}(3)$	3^2	[Der]	$\mathrm{C}_n{}^{\mathrm{ad}}(2)$ $(n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$\mathrm{G_2}(3)$	$1^6, 3^2$	[C]	$\mathrm{C}_n{}^{\mathrm{sc}}(2)$ $(n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	[$\mathrm{B_2}^{\mathrm{sc}}$]
$\mathrm{A}_3^{\mathrm{sc},(2)}(2)$	4^3	[Der]	$\mathrm{D}_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$\mathrm{B_2}^{\mathrm{ad}}(2)$	$2^2, 4$	[C]	$\mathrm{D_4}^{\mathrm{sc}}(2)$	8^3	[Der]
$\mathrm{B}_n{}^{\mathrm{ad}}(2)$ $(n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$\mathrm{D}_n^{(1)}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_2}^{\mathrm{sc}}(2)$	$\mathbf{4}, 4$	[$\mathrm{B_2}^{\mathrm{sc}}$]	$\mathrm{D}_n{}^{\mathrm{sc}}(2)$ $(n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_3}^{\mathrm{sc}}(2)$	6^3	[Der]	$\mathrm{F_4}(2)$	$2^{12}, 8^3$	[C]
$\mathrm{B_4}^{\mathrm{sc}}(2)$	$2^4, 8^3$	[Der]	$\mathrm{G_2}(2)$	4^3	[Der]
$\mathrm{B}_n{}^{\mathrm{sc}}(2)$ $(n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	[$\mathrm{A_2}$]

Table 1. Multidimensional root spaces

Diagonalising (overview)

$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$\mathrm{A_2}^{\mathrm{sc}}(3)$	3^2	[Der]	$\mathrm{C}_n{}^{\mathrm{ad}}(2)\ (n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$\mathrm{G_2}(3)$	$1^6, 3^2$	[C]	$\mathrm{C}_n{}^{\mathrm{sc}}(2)\ (n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	[$\mathrm{B_2}^{\mathrm{sc}}$]
$\mathrm{A}_3^{\mathrm{sc},(2)}(2)$	4^3	[Der]	$\mathrm{D}_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$\mathrm{B_2}^{\mathrm{ad}}(2)$	$2^2, 4$	[C]	$\mathrm{D_4}^{\mathrm{sc}}(2)$	8^3	[Der]
$\mathrm{B}_n{}^{\mathrm{ad}}(2)\ (n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$\mathrm{D}_n^{(1)}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_2}^{\mathrm{sc}}(2)$	$\mathbf{4}, 4$	[$\mathrm{B_2}^{\mathrm{sc}}$]	$\mathrm{D}_n{}^{\mathrm{sc}}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_3}^{\mathrm{sc}}(2)$	6^3	[Der]	$\mathrm{F_4}(2)$	$2^{12}, 8^3$	[C]
$\mathrm{B_4}^{\mathrm{sc}}(2)$	$2^4, 8^3$	[Der]	$\mathrm{G_2}(2)$	4^3	[Der]
$\mathrm{B}_n{}^{\mathrm{sc}}(2)\ (n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	[$\mathrm{A_2}$]

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Diagonalising (overview)

$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
$\mathrm{A_2}^{\mathrm{sc}}(3)$	3^2	[Der]	$\mathrm{C}_n{}^{\mathrm{ad}}(2)\ (n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$\mathrm{G_2}(3)$	$1^6, 3^2$	[C]	$\mathrm{C}_n{}^{\mathrm{sc}}(2)\ (n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	$[\mathrm{B_2}^{\mathrm{sc}}]$
$\mathrm{A}_3^{\mathrm{sc},(2)}(2)$	4^3	[Der]	$\mathrm{D}_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
$\mathrm{B_2}^{\mathrm{ad}}(2)$	$2^2, 4$	[C]	$\mathrm{D_4}^{\mathrm{sc}}(2)$	8^3	[Der]
$\mathrm{B}_n{}^{\mathrm{ad}}(2)\ (n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$\mathrm{D}_n^{(1)}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_2}^{\mathrm{sc}}(2)$	$\mathbf{4}, 4$	$[\mathrm{B_2}^{\mathrm{sc}}]$	$\mathrm{D}_n{}^{\mathrm{sc}}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
$\mathrm{B_3}^{\mathrm{sc}}(2)$	6^3	[Der]	$\mathrm{F_4}(2)$	$2^{12}, 8^3$	[C]
$\mathrm{B_4}^{\mathrm{sc}}(2)$	$2^4, 8^3$	[Der]	$\mathrm{G_2}(2)$	4^3	[Der]
$\mathrm{B}_n{}^{\mathrm{sc}}(2)\ (n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	$[\mathrm{A_2}]$

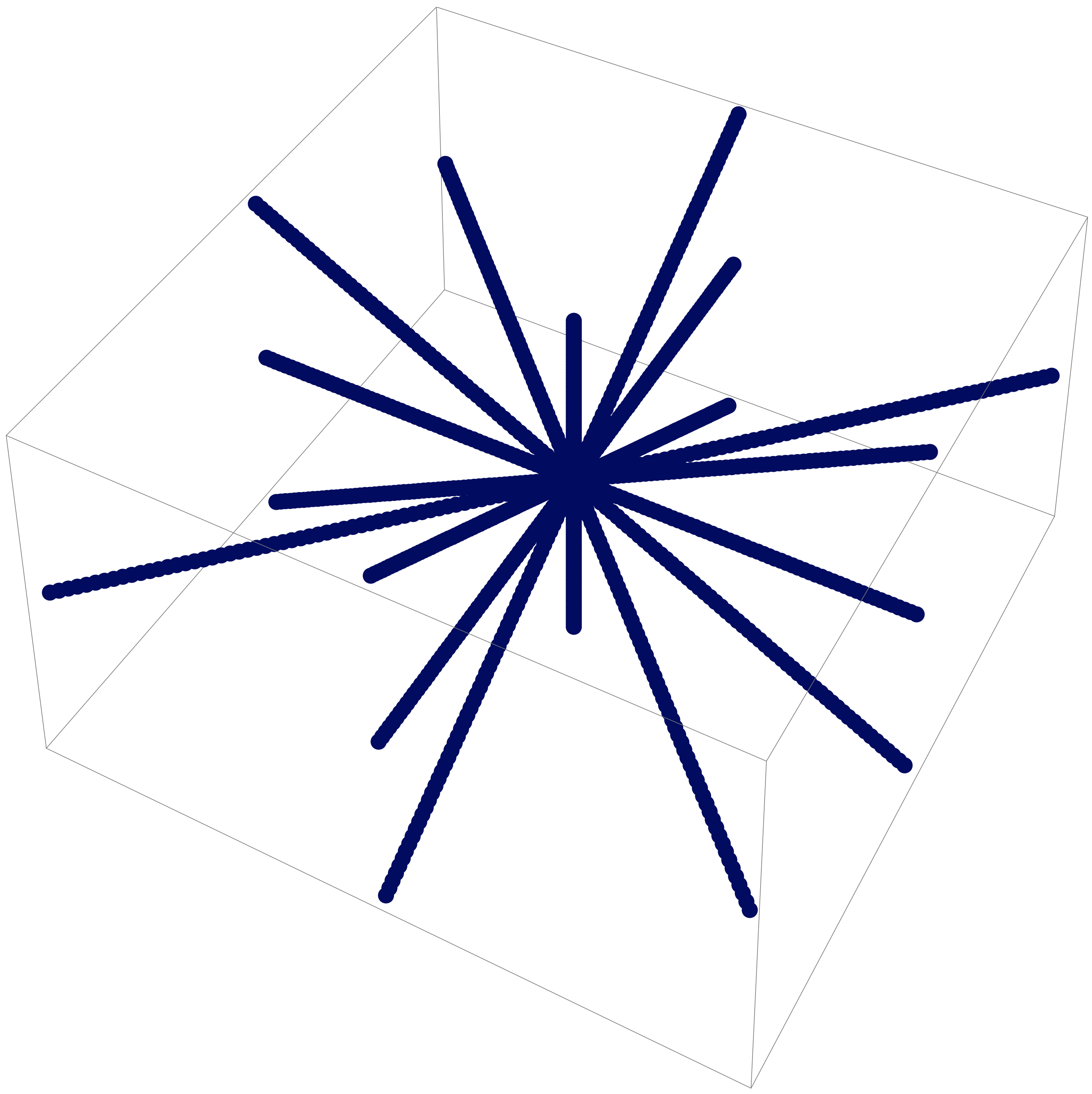
Table 1. Multidimensional root spaces

Diagonalising (overview)

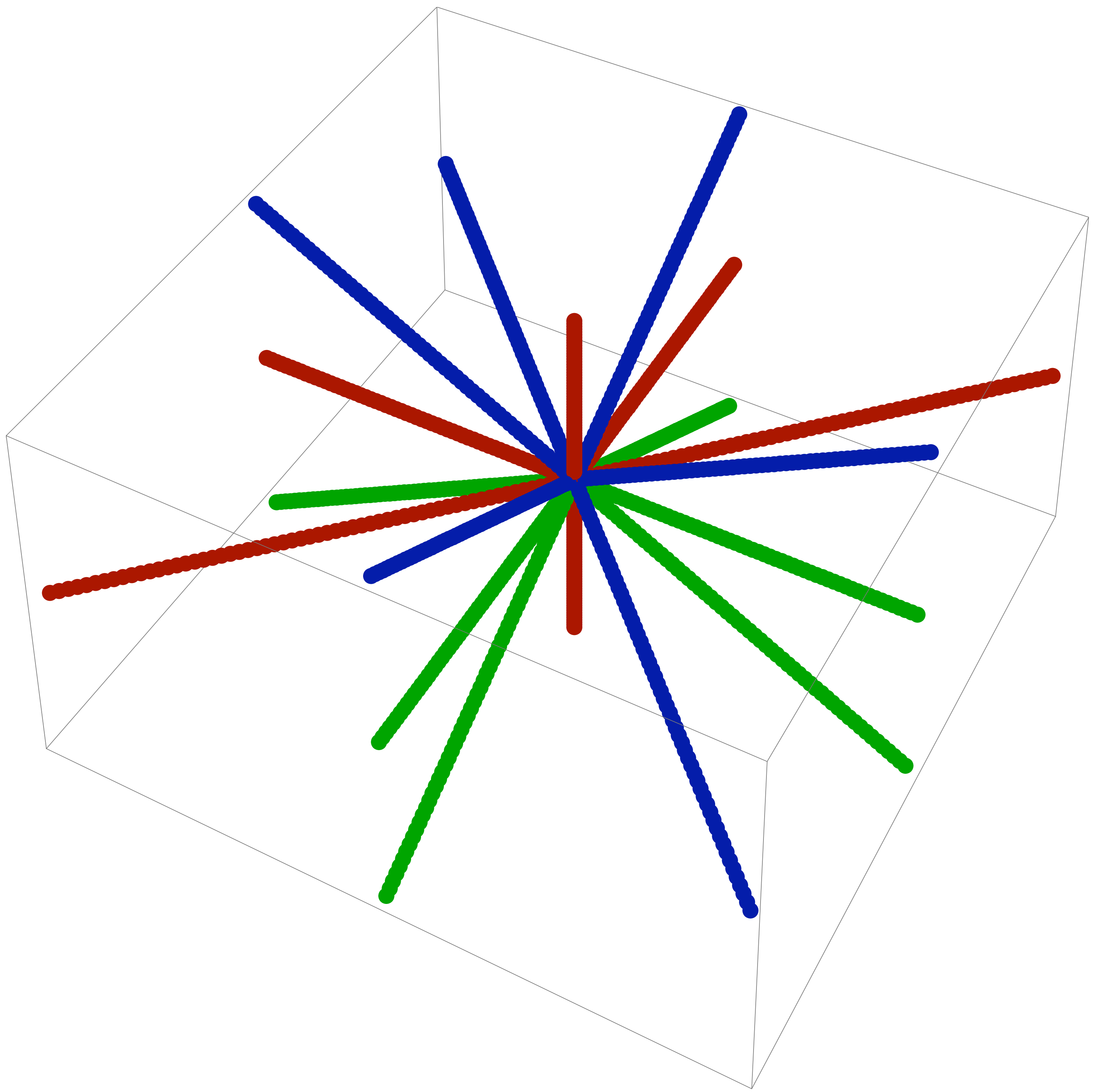
$R(p)$	Mults	Soln	$R(p)$	Mults	Soln
${A_2}^{sc}(3)$	3^2	[Der]	${C_n}^{ad}(2)\ (n \geq 3)$	$2n, 2^{n(n-1)}$	[C]
$G_2(3)$	$1^6, 3^2$	[C]	${C_n}^{sc}(2)\ (n \geq 3)$	$\mathbf{2n}, 4^{\binom{n}{2}}$	$[{B_2}^{sc}]$
$A_3^{sc,(2)}(2)$	4^3	[Der]	$D_4^{(1),(n-1),(n)}(2)$	4^6	[Der]
${B_2}^{ad}(2)$	$2^2, 4$	[C]	${D_4}^{sc}(2)$	8^3	[Der]
${B_n}^{ad}(2)\ (n \geq 3)$	$2^n, 4^{\binom{n}{2}}$	[C]	$D_n^{(1)}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
${B_2}^{sc}(2)$	$\mathbf{4}, 4$	$[{B_2}^{sc}]$	${D_n}^{sc}(2)\ (n \geq 5)$	$4^{\binom{n}{2}}$	[Der]
${B_3}^{sc}(2)$	6^3	[Der]	$F_4(2)$	$2^{12}, 8^3$	[C]
${B_4}^{sc}(2)$	$2^4, 8^3$	[Der]	$G_2(2)$	4^3	[Der]
${B_n}^{sc}(2)\ (n \geq 5)$	$2^n, 4^{\binom{n}{2}}$	[C]	all remaining(2)	$2^{\lvert\Phi^+\rvert}$	$[A_2]$

TABLE 1. Multidimensional root spaces

Diagonalising (B_3, char. 2)



Diagonalising (B_3, char. 2)



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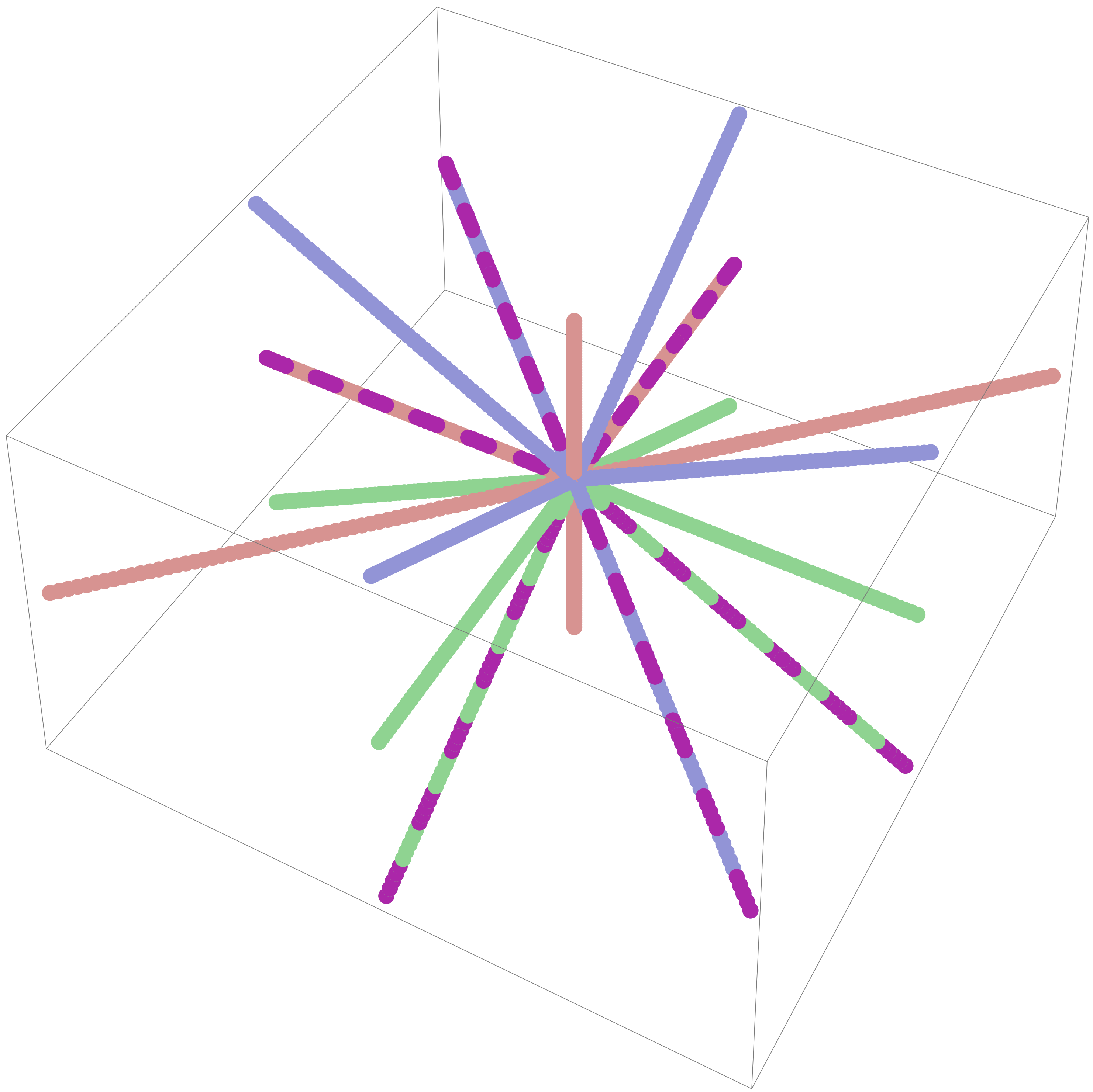
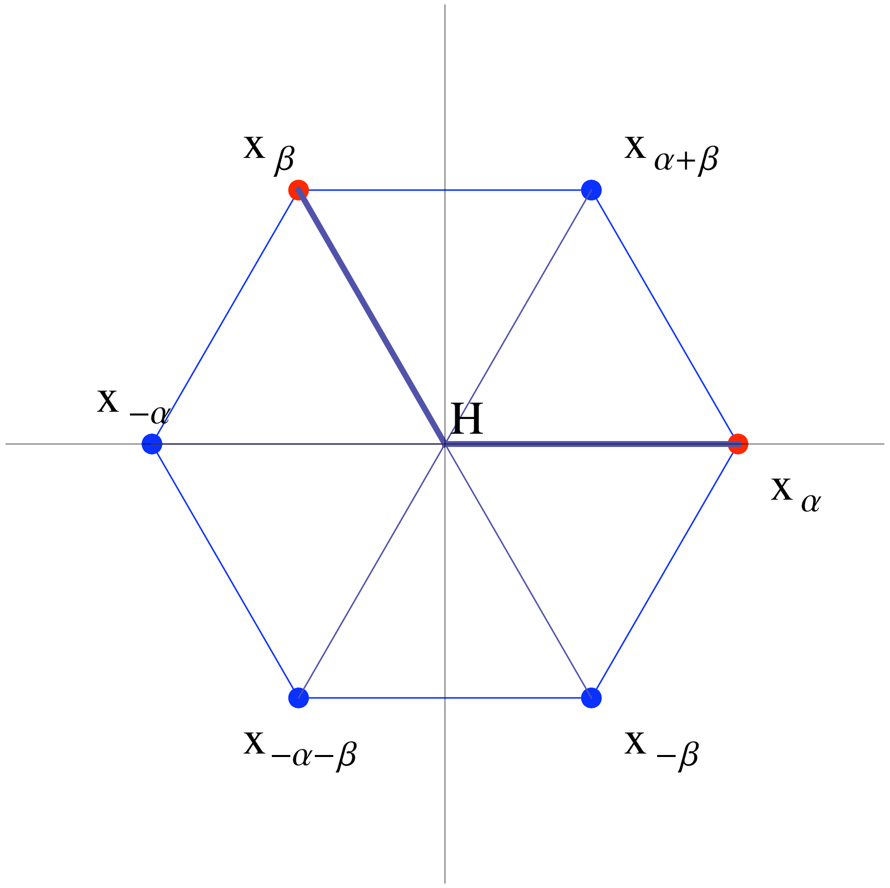
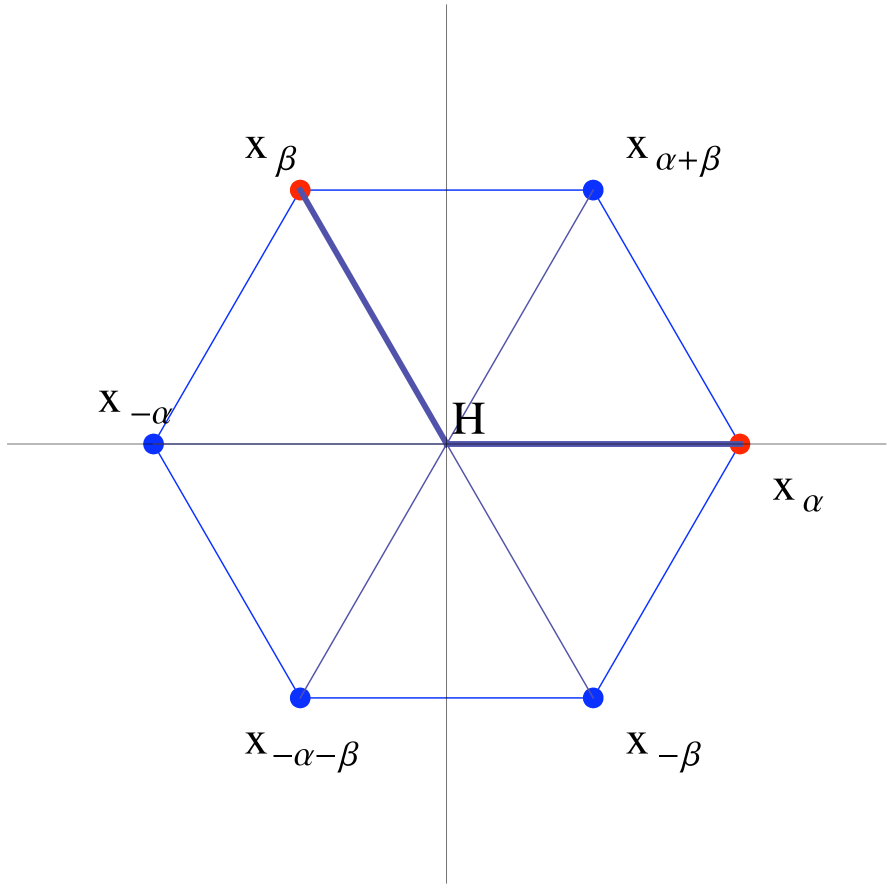


Diagonalising (A_2, char. 3)



Type	Eigenspaces	Composition
Ad	(2,) 1^6	$\frac{1}{7}$
SC	(2,) 3^2	$\frac{7}{1}$

Diagonalising (A_2, char. 3)



Type	Eigenspaces	Composition
Ad	(2,) 1^6	$\frac{1}{7}$
SC	(2,) 3^2	$\frac{7}{1}$

Observations:

- There is only one "7",
- $\mathrm{Der}(L^{SC}) = L^{Ad}$.

Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- Does it work?
- What next?

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A tiny demo: A_3 / sl_4

A tiny demo: A_3 / sl_4


```
danroozemond@dyn183 ~ $ magma-exp
Magma V2.15-9     Wed May 13 2009 10:38:45 on dyn183   [Seed = 2715905262]
Type ? for help.  Type <Ctrl>-D to quit.

Loading startup file "/Users/danroozemond/.magmarc"

======Warning! 1500 M memory limit active.======
[~/tue/research/cb/magma-pkg/all.spec attached]
> //Construct sl_4 over the rationals
> Q := Rationals();
> gl4Q := MatrixLieAlgebra(Q, 4);
> Dimension(gl4Q);
16
>
> sl4Q := sub<gl4
```


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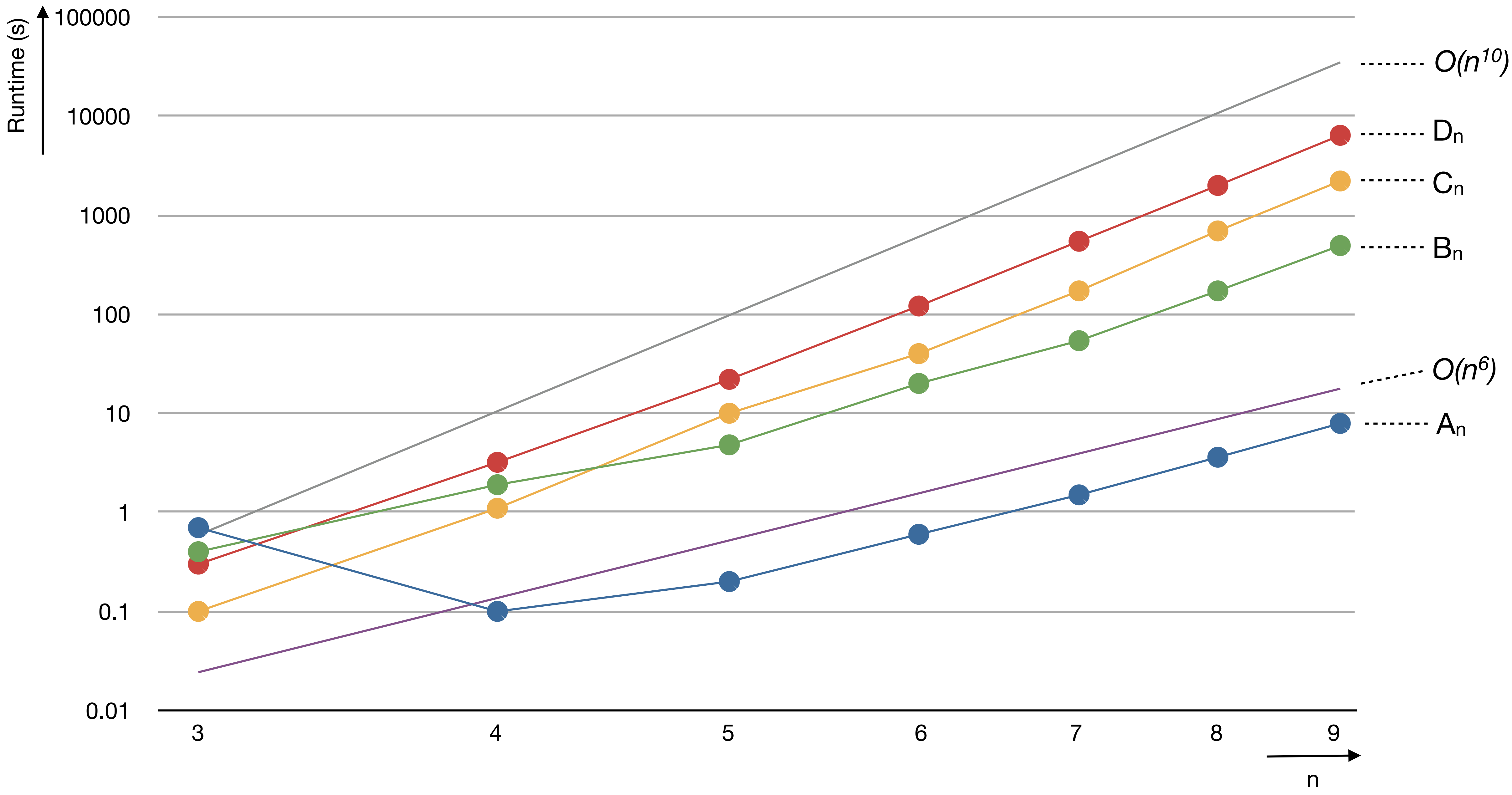
A graph



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Conclusion

- Main challenges for computing Chevalley bases in small characteristic:
  - Multidimensional eigenspaces,

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Conclusion

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  - Broken root chains;

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- Found solutions for all cases,
  - and implemented these in MAGMA;

Conclusion

- Main challenges for computing Chevalley bases in small characteristic:
  - Multidimensional eigenspaces,
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- To do:
  - Compute split Cartan subalgebras in small characteristic;

Conclusion

- Main challenges for computing Chevalley bases in small characteristic:
  - Multidimensional eigenspaces,
  - Broken root chains;
- Found solutions for all cases,
  - and implemented these in MAGMA;
- To do:
  - Compute split Cartan subalgebras in small characteristic;
- Bigger picture:
  - Recognition of groups or Lie algebras,
  - Finding conjugators for Lie group elements,
  - Finding automorphisms of Lie algebras,
  - ...

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Outline

- What is a Lie algebra?
- What is a Chevalley basis?
- How to compute Chevalley bases?
- Does it work?
- What next?

- **Any questions?**

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