Construction of Chevalley Bases of Lie Algebras

Dan Roozemond
Joint work with Arjeh M. Cohen

December 3rd, 2008, EIDMA Combinatorial Seminar,
Eindhoven University of Technology

http://www.win.tue.nl/∼droozemo/ (or Google)

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Outline

1. Why study Lie algebras?
2. Defining Lie algebras
    - Root system
    - Root datum
    - Lie algebra
3. Examples
    - A_1, B_2, G_2
    - A_2
4. Computing Chevalley Bases
    - Why?
    - How?
    - Strange things in small characteristic
    - Solving these things
5. Conclusion, Future research

Why Study Lie Algebras?

- Study groups by Lie algebras:
  - Simple algebraic group $G \leftrightarrow$ Unique Lie algebra L
  - Many properties carry over to L
  - Easier to calculate in L
  - $G \leq \mathrm{Aut}(L)$, often even $G = \mathrm{Aut}(L)$
- Opportunities for:
  - Recognition
  - Conjugation
  - ...
- Because there are problems to be solved!

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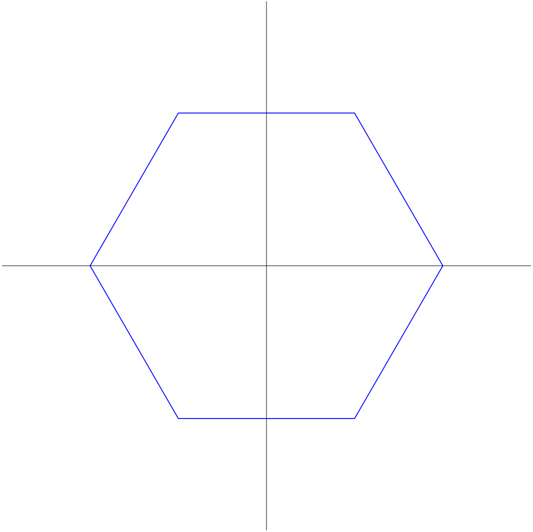
Root Systems



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- A root system of type A_2
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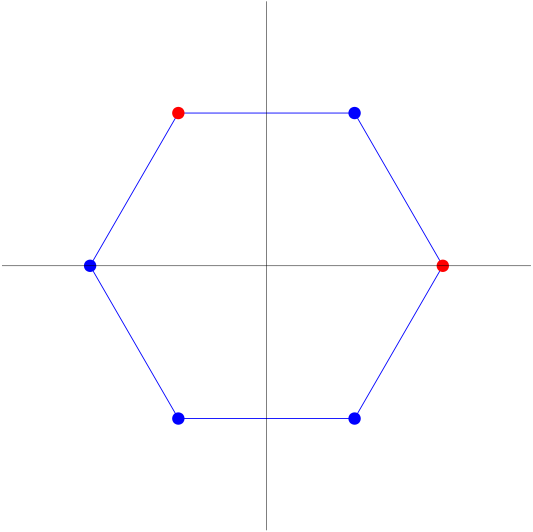

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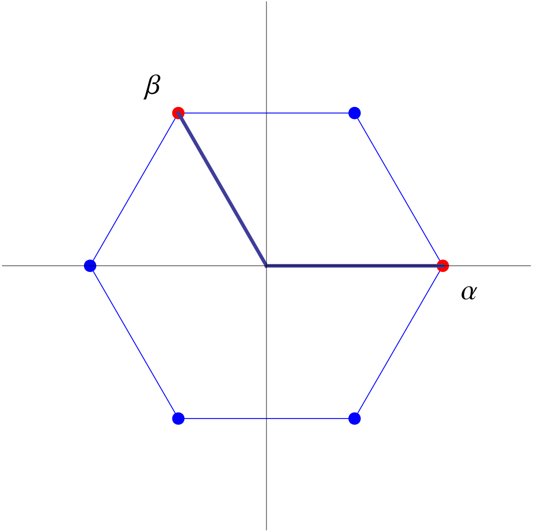

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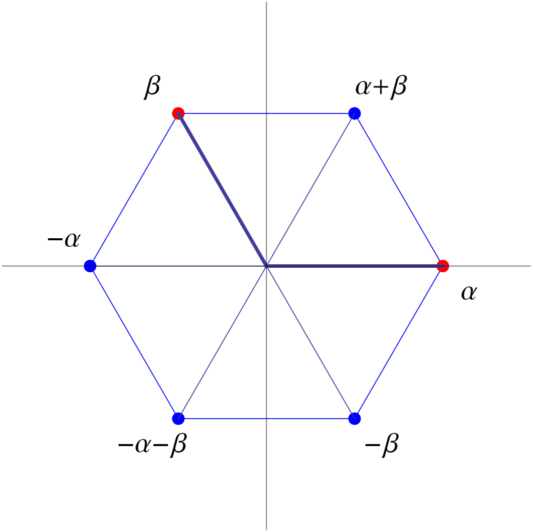

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Root Data

Definition (Root Datum)

$$R = (X, \Phi, Y, \Phi^\vee), \quad \langle\cdot,\cdot\rangle : X \times Y \to \mathbb{Z},$$

- X, Y: dual free $\mathbb{Z}$-modules,
- put in duality by $\langle\cdot,\cdot\rangle$,
- $\Phi \subseteq X$: roots,
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One Root System $\rightarrow$ { Several Root Data: "adjoint" ⋮ "simply connected" }

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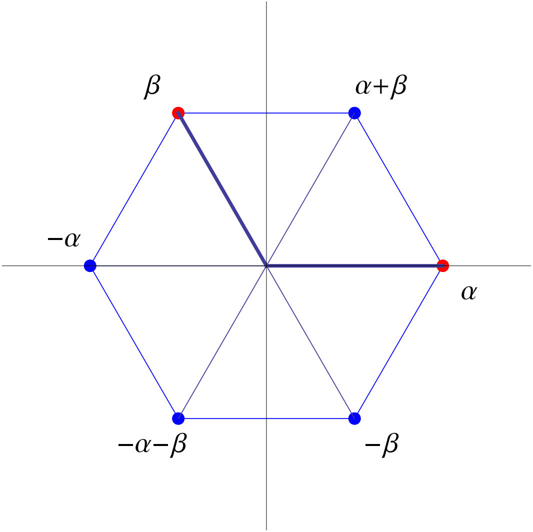

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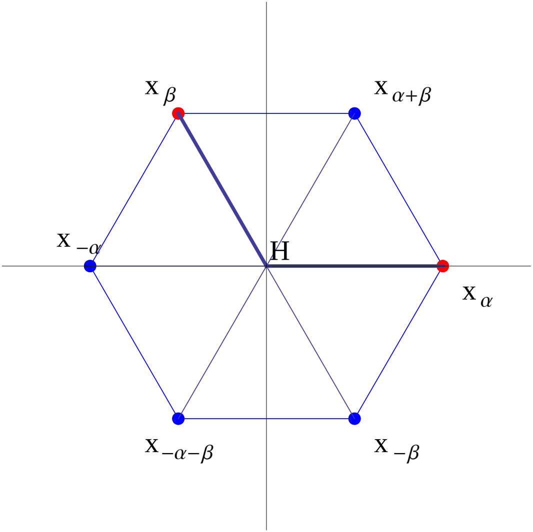

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Definition (Chevalley Lie Algebra)

$$\text{Formal basis}: L_{\mathbb{Z}} = \bigoplus_{i=1,\dots,n} \mathbb{Z}h_i \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}x_\alpha,$$

$$\text{Multiplication}: [\cdot, \cdot]$$

with bilinear antisymmetric multiplication defined by

- $h_i, h_j \in H: \quad [h_i, h_j] = 0,$
- $h_i \in H, \alpha \in \Phi: \quad [x_\alpha, h_i] = \langle \alpha, f_i \rangle X_\alpha,$
- $\alpha \in \Phi: \quad [x_{-\alpha}, x_\alpha] = \sum_{i=1}^n \langle e_i, \alpha^\vee \rangle h_i,$
- $\alpha, \beta \in \Phi: \quad [x_\alpha, x_\beta] = \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$
- $+$ Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

Such a basis: a *Chevalley basis*. $N_{\alpha,\beta} = \pm(k+1)$

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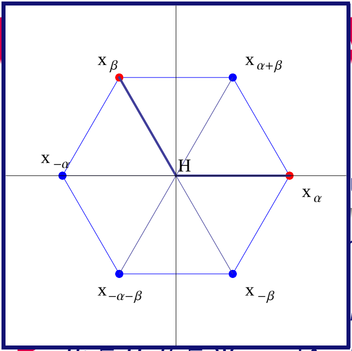

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$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

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$\mathfrak{sl}_2$: Trace 0 matrices.

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
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$$[a, b] := ab - ba$$

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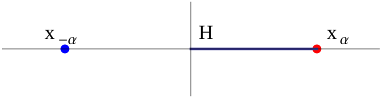

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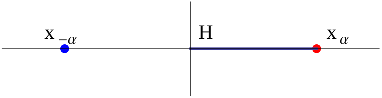

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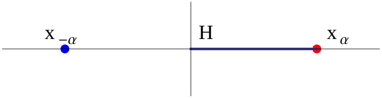

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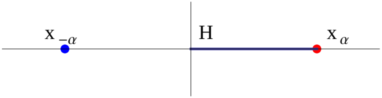

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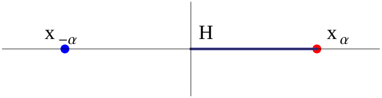

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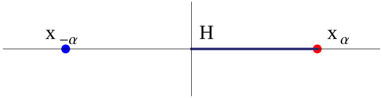

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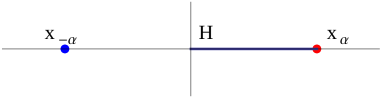

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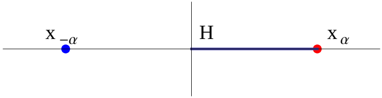

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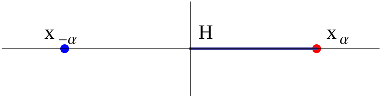

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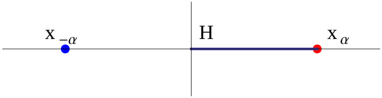

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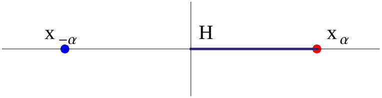

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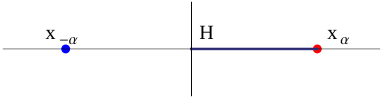

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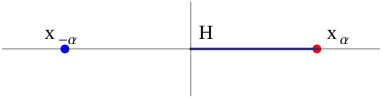

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$$\begin{aligned} L_\mathbb{Z} &= H \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}x_\alpha \\ &= \mathbb{Z}h \oplus \mathbb{Z}x_\alpha \oplus \mathbb{Z}x_{-\alpha}, \end{aligned}$$

	x_α	$x_{-\alpha}$	h
x_α	0	$-2h$	x_α
$x_{-\alpha}$	$2h$	0	$-x_{-\alpha}$
h	$-x_\alpha$	$x_{-\alpha}$	0

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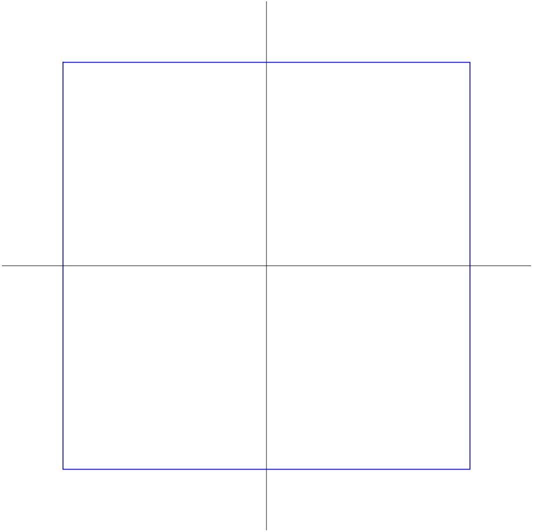
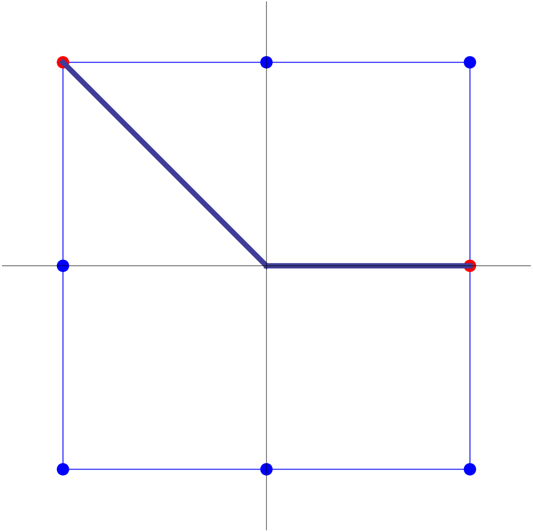
Example: B_2



- A square
- A root system of type B_2
- A Lie algebra of type B_2

Example: B_2



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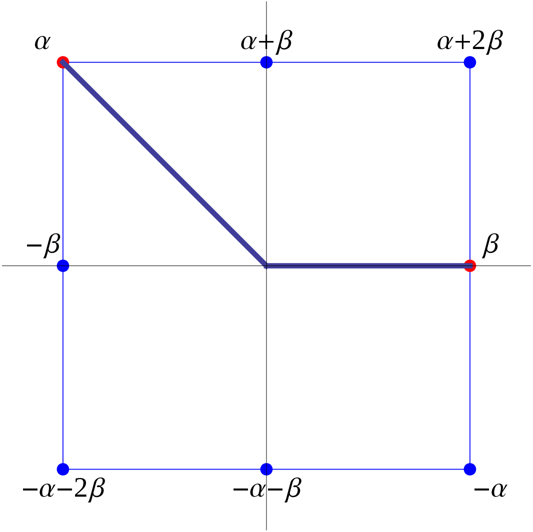

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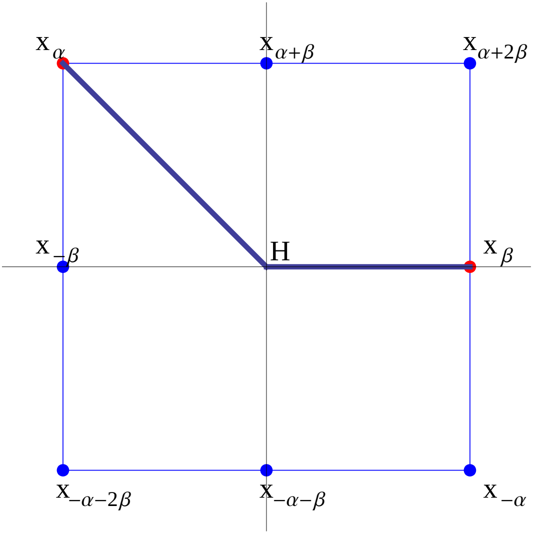

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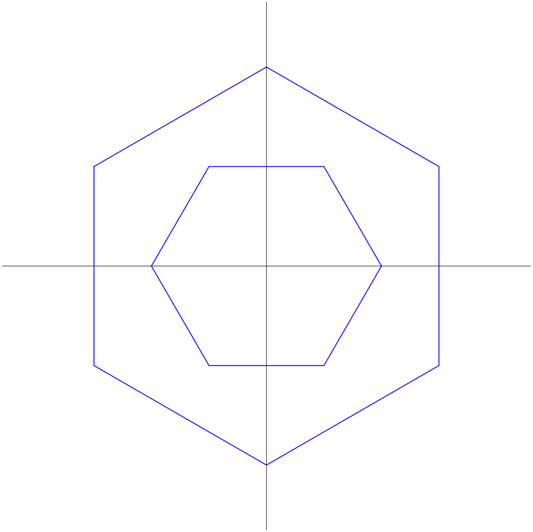
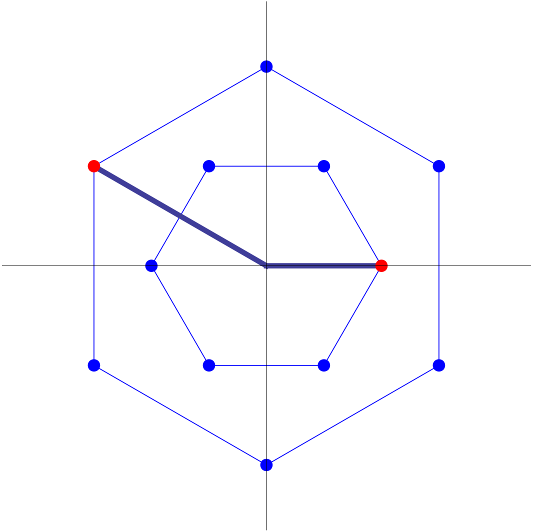
Example: G_2



- Two hexagons
- A root system of type G_2
- A Lie algebra of type G_2

Example: G_2



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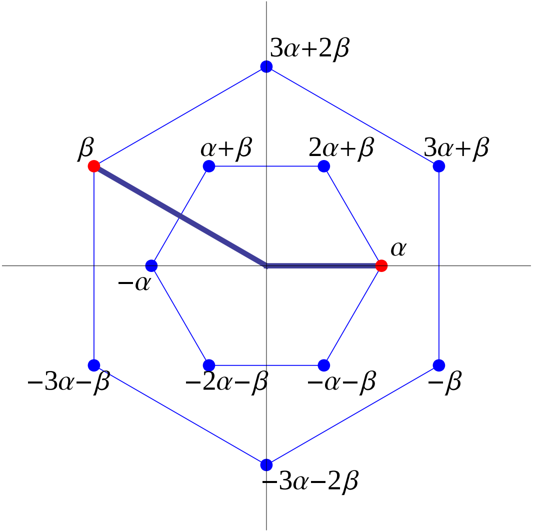
Example: G_2



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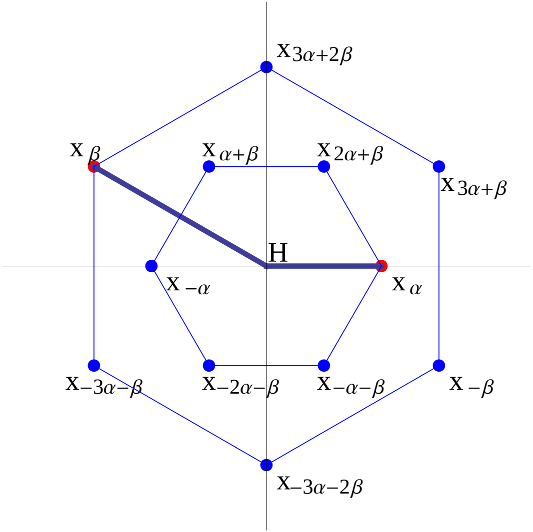
Example: G_2



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Outline

1. Why study Lie algebras?
2. Defining Lie algebras
    - Root system
    - Root datum
    - Lie algebra
3. Examples
    - A_1, B_2, G_2
    - A_2
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    - How?
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    - Solving these things
5. Conclusion, Future research

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Big example: 3×3 matrices, trace 0

- L = matrices, 3×3, trace 0;
- $[x, y] := xy - yx$;
- $$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$
- Claim: L is of type A_2.

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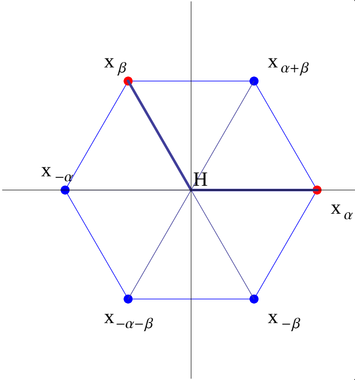
Big example: 3×3 matrices, trace 0 (contd)

- $h_i, h_j \in H$: $[h_i, h_j] = 0,$
- $h_i \in H, \alpha \in \Phi$: $[x_\alpha, h_i] = \langle \alpha, f_i \rangle X_\alpha,$
- $\alpha \in \Phi$: $[x_{-\alpha}, x_\alpha] = \sum_{i=1}^{n} \langle e_i, \alpha^\vee \rangle h_i,$
- $\alpha, \beta \in \Phi$: $[x_\alpha, x_\beta] = \begin{cases} N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$

"Adjoint" root datum:

- Pos. roots: $(1, 0), (0, 1), (1, 1),$
- Pos. coroots: $(2, -1), (-1, 2), (1, 1).$

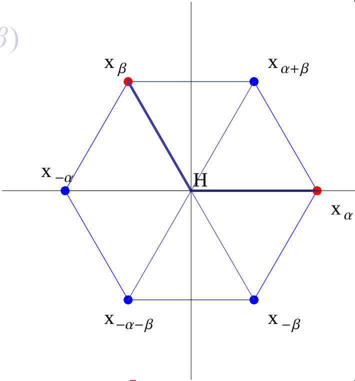


Big example: 3×3 matrices, trace 0 (contd)

- So we can compute ~~a Chevalley basis~~ Chevalley bases in this case!
- And thus exhibit a (quite special) element of Aut(L):

$$\begin{array}{rcl} \alpha & \leftrightarrow & -\alpha \\ \beta & \leftrightarrow & \alpha+\beta \\ -\beta & \leftrightarrow & -(\alpha+\beta) \end{array}$$

- Can we make the machine do this?



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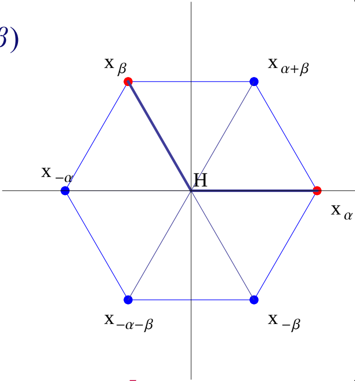

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Chevalley Bases

Definition (Chevalley Lie Algebra)

$$\text{Formal basis} : L_{\mathbb{Z}} = \bigoplus_{i=1,\dots,n} \mathbb{Z} h_i \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z} x_\alpha ,$$

$$\text{Multiplication} : [\cdot, \cdot]$$

$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

- Idea: Given any Lie algebra, find a Chevalley basis.
- Why?
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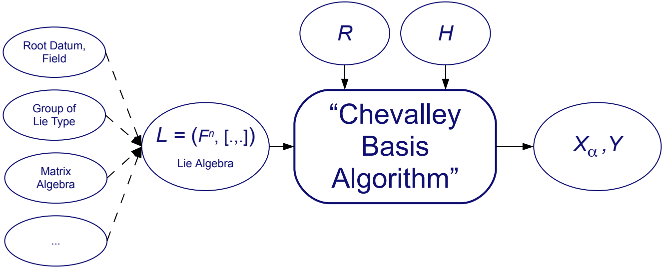

(Cohen/Murray, indep. Ryba)

Also given: Root datum R, splitting Cartan subalgebra $H = Y \otimes \mathbb{F}$

(De Graaf, Murray)

Char. 0, p $\geq$ 5: Implemented in GAP, Magma

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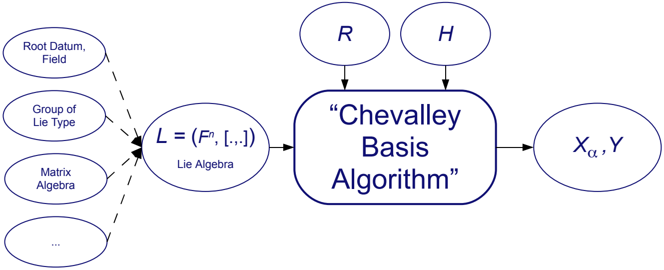
The Game



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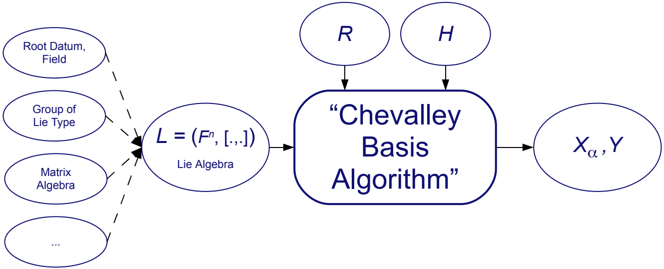

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Chevalley Basis Algorithm

ChevalleyBasis

in: A simple Lie algebra L,
a splitting Cartan subalgebra H of L, and
a root datum $R = (X, \Phi, Y, \Phi^\vee)$.

out: A Chevalley basis B for L with respect to H and R.

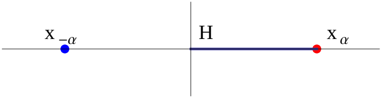
begin

let $\{E_1, \dots, E_m\} = \textsc{Diagonalize}(L, H)$,
let $\{\overline{X}_1, \dots, \overline{X}_{|\Phi|}\} = \textsc{Straighten}(L, \{E_1, \dots, E_m\})$,
let $\iota = \textsc{IdentifyRoots}(L, R, \{\overline{X}_1, \dots, \overline{X}_{|\Phi|}\})$,
let $[X_\alpha \mid \alpha \in \Phi], [h_1, \dots, h_{\mathrm{rnk}(\Phi)}] = \textsc{ScaleToBasis}(L, H, \{\overline{X}_1, \dots, \overline{X}_{|\Phi|}\}, \iota)$,
return $[X_\alpha \mid \alpha \in \Phi], [h_1, \dots, h_{\mathrm{rnk}(\Phi)}]$.

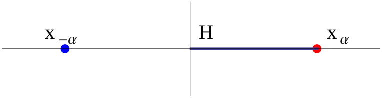
end

Algorithm: Finding a Chevalley Basis

Strange things in small characteristic (I)



	x_α	$x_{-\alpha}$	h
x_α	0	$-h$	$2x_\alpha$
$x_{-\alpha}$	h	0	$-2x_{-\alpha}$
h	$-2x_\alpha$	$2x_{-\alpha}$	0



	x_α	$x_{-\alpha}$	h
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h	$-x_\alpha$	$x_{-\alpha}$	0

Observe:

- $h \mapsto \frac{1}{2}h$ maps $\mathrm{Lie}({A_1}^{sc}, \mathbb{F})$ to $\mathrm{Lie}({A_1}^{ad}, \mathbb{F})$,
- So $\mathrm{Lie}({A_1}^{sc}, \mathbb{F}) \cong \mathrm{Lie}({A_1}^{ad}, \mathbb{F})$,
- Except if $\mathrm{char}(\mathbb{F}) = 2$!

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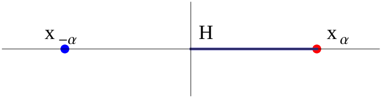

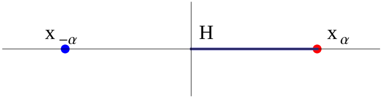

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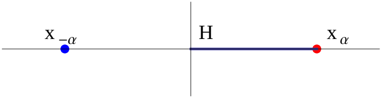

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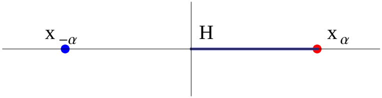

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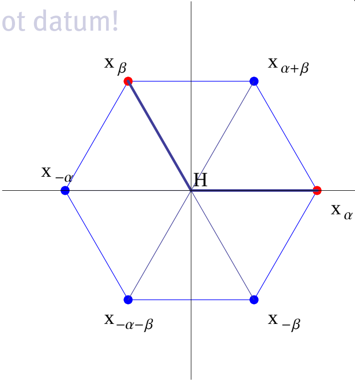
Strange things in small characteristic (II)

- $h_1 = -\frac{2}{3}c_1 - \frac{1}{3}c_2$
  $h_2 = -\frac{1}{3}c_1 - \frac{2}{3}c_2$
- But then what happens in char. 3 ?!
- We computed with the "adjoint" root datum; but Trace 0 matrices $\leftrightarrow$ "simply connected" root datum!
- Isomorphic $\Leftrightarrow$ char. $\neq 3$!



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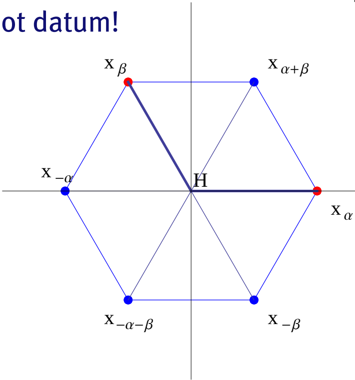

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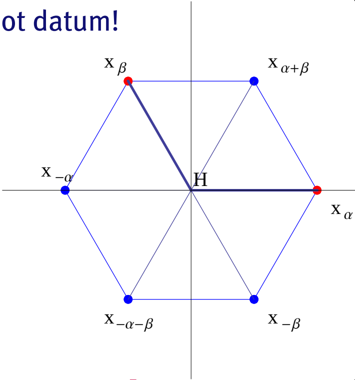
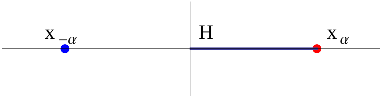
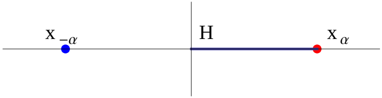


Strange things in small characteristic (III)



	x_α	$x_{-\alpha}$	h	$\mathbb{Z}^1$
x_α	0	$-h$	$2x_\alpha$	(2)
$x_{-\alpha}$	h	0	$-2x_{-\alpha}$	(-2)
h	$-2x_\alpha$	$2x_{-\alpha}$	0	(0)



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- Use action of H to diagonalize L and find x_α,
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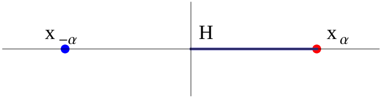
Strange things in small characteristic (III)



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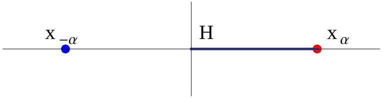

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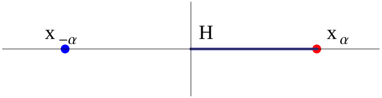

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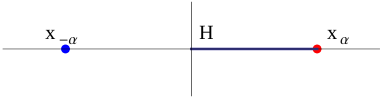

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Strange things in small characteristic (IV)

	...	h_1	h_2	$\mathbb{Z}$
x_α		$2x_\alpha$	$-x_\alpha$	$(2, -1)$
x_β		$-3x_\beta$	$2x_\beta$	$(-3, 2)$
$x_{\alpha+\beta}$		$-x_{\alpha+\beta}$	$x_{\alpha+\beta}$	$(-1, 1)$
$x_{2\alpha+\beta}$		$x_{2\alpha+\beta}$	0	$(1, 0)$
$x_{3\alpha+\beta}$		$3x_{3\alpha+\beta}$	$-x_{3\alpha+\beta}$	$(3, -1)$
$x_{3\alpha+2\beta}$		0	$x_{3\alpha+2\beta}$	$(0, 1)$
$\vdots$				

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Strange things in small characteristic (IV)

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$x_{2\alpha+\beta}$		$x_{2\alpha+\beta}$	0	$(1,0)$
$x_{3\alpha+\beta}$		$3x_{3\alpha+\beta}$	$-x_{3\alpha+\beta}$	$(3,-1)$
$x_{3\alpha+2\beta}$		0	$x_{3\alpha+2\beta}$	$(0,1)$
$\vdots$				

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$x_{3\alpha+\beta}$		$3x_{3\alpha+\beta}$	$-x_{3\alpha+\beta}$	$(3,-1)$
$x_{3\alpha+2\beta}$		0	$x_{3\alpha+2\beta}$	$(0,1)$
$x_{-\alpha}$		$-2x_{-\alpha}$	$x_{-\alpha}$	$(-2,1)$
$x_{-\beta}$		$3x_{-\beta}$	$-2x_{-\beta}$	$(3,-2)$
$x_{-\alpha-\beta}$		$x_{-\alpha-\beta}$	$-x_{-\alpha-\beta}$	$(1,-1)$
$x_{-2\alpha-\beta}$		$-x_{-2\alpha-\beta}$	0	$(-1,0)$
$x_{-3\alpha-\beta}$		$-3x_{-3\alpha-\beta}$	$x_{-3\alpha-\beta}$	$(-3,1)$
$x_{-3\alpha-2\beta}$		0	$-x_{-3\alpha-2\beta}$	$(0,-1)$
$\vdots$				

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Strange things in small characteristic (IV)

	…	h_1	h_2	$\mathbb{Z}$	GF(3^m)	
x_α		$2x_\alpha$	$-x_\alpha$	$(2,-1)$	$(-1,-1)$	
x_β		$-3x_\beta$	$2x_\beta$	$(-3,2)$	$(0,-1)$	(!)
$x_{\alpha+\beta}$		$-x_{\alpha+\beta}$	$x_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$x_{2\alpha+\beta}$		$x_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$x_{3\alpha+\beta}$		$3x_{3\alpha+\beta}$	$-x_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$x_{3\alpha+2\beta}$		0	$x_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	
$x_{-\alpha}$		$-2x_{-\alpha}$	$x_{-\alpha}$	$(-2,1)$	$(1,1)$	
$x_{-\beta}$		$3x_{-\beta}$	$-2x_{-\beta}$	$(3,-2)$	$(0,1)$	
$x_{-\alpha-\beta}$		$x_{-\alpha-\beta}$	$-x_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$x_{-2\alpha-\beta}$		$-x_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$x_{-3\alpha-\beta}$		$-3x_{-3\alpha-\beta}$	$x_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	
$x_{-3\alpha-2\beta}$		0	$-x_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
⋮						

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Strange things in small characteristic (IV)

	…	h_1	h_2	$\mathbb{Z}$	$GF(3^m)$	
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$x_{2\alpha+\beta}$		$x_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$x_{3\alpha+\beta}$		$3x_{3\alpha+\beta}$	$-x_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$x_{3\alpha+2\beta}$		0	$x_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	$(!^2)$
$x_{-\alpha}$		$-2x_{-\alpha}$	$x_{-\alpha}$	$(-2,1)$	$(1,1)$	
$x_{-\beta}$		$3x_{-\beta}$	$-2x_{-\beta}$	$(3,-2)$	$(0,1)$	$(!^2)$
$x_{-\alpha-\beta}$		$x_{-\alpha-\beta}$	$-x_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
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⋮						

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Multidimensional Eigenspaces

Char.	Root datum	Eigenspace dims
3	$A_2{}^{sc}$	3^2
3	G_2	$1^6, 3^2$
2	$A_3{}^{sc}, A_3^{(a)\star}$	4^3
2	$B_n{}^{ad}$ $(n \geq 2)$	$2^n, 4^{\binom{n}{2}}$
2	$B_2{}^{sc}$	4^2
2	$B_3{}^{sc}$	6^3
2	$B_4{}^{sc}$	$2^4, 8^3$
2	$B_n{}^{sc}$ $(n \geq 5)$	$2^n, 4^{\binom{n}{2}}$
2	$C_n{}^{ad}$ $(n \geq 3)$	$2n^1, 2^{2\binom{n}{2}}$
2	$C_n{}^{sc}$ $(n \geq 3)$	$2n^1, 4^{\binom{n}{2}}$
2	$D_4^{(a),(b),(a+b)\star}$	4^6
2	$D_4{}^{sc}$	8^3
2	$D_n^{(a)\star}, D_n{}^{sc}$ $(n \geq 5)$	$4^{\binom{n}{2}}$
2	F_4	$2^{12}, 8^3$
2	G_2	4^3
2	all remaining cases	$2^N (N = \|\Phi^+\|)$

Steinberg, 1961

Complete list of multiplicities of roots, for root data of adjoint type

Cohen, R., 2008

Complete list of multiplicities of roots, for all root data

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Outline

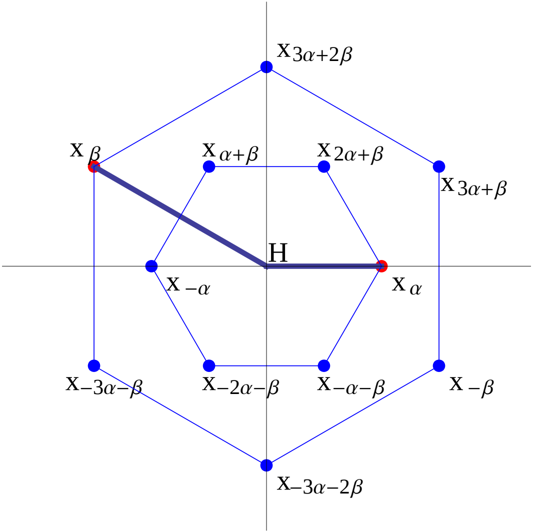
1. Why study Lie algebras?
2. Defining Lie algebras
   - Root system
   - Root datum
   - Lie algebra
3. Examples
   - A_1, B_2, G_2
   - A_2
4. Computing Chevalley Bases
   - Why?
   - How?
   - Strange things in small characteristic
   - Solving these things
5. Conclusion, Future research

Multidimensional Eigenspaces

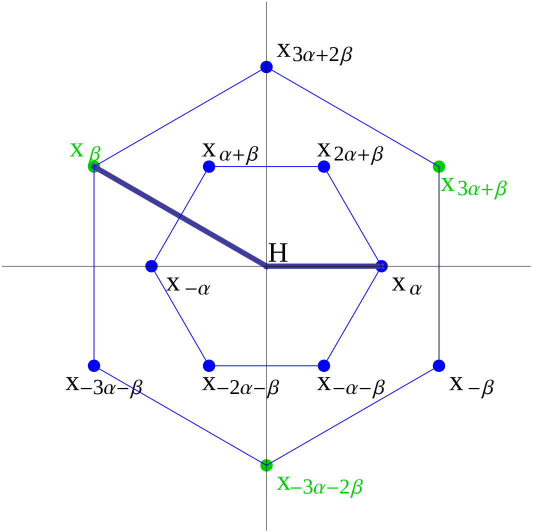
General Solution Strategies:

1. Nullspaces (ex: G_2, char. 3),
2. Ideals (ex: B_3, char. 2),
3. Derivation Algebra (ex: A_2, char. 3)

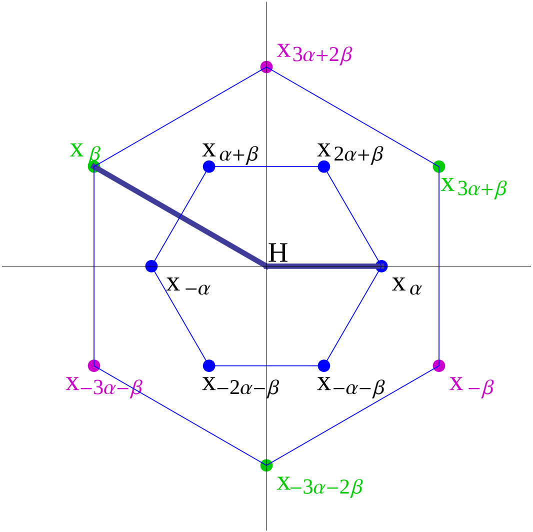
Example: Solving G_2 in char. 3



Example: Solving G_2 in char. 3



Example: Solving G_2 in char. 3



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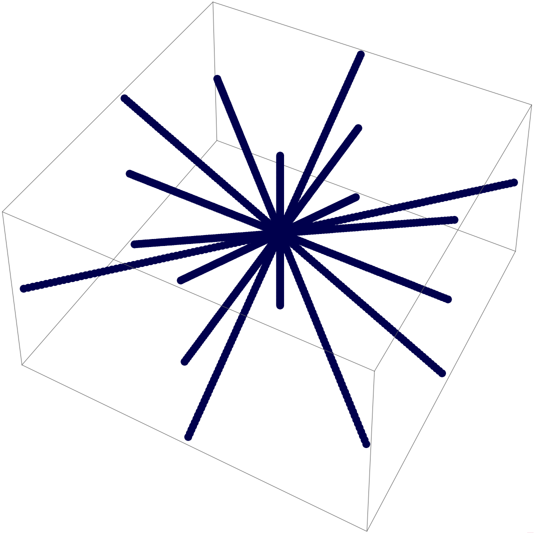
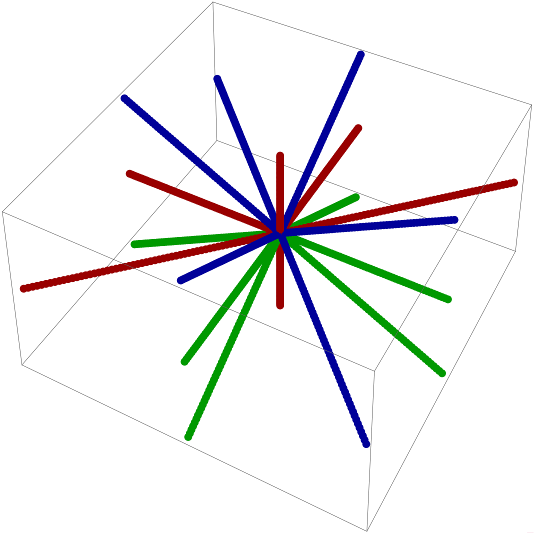
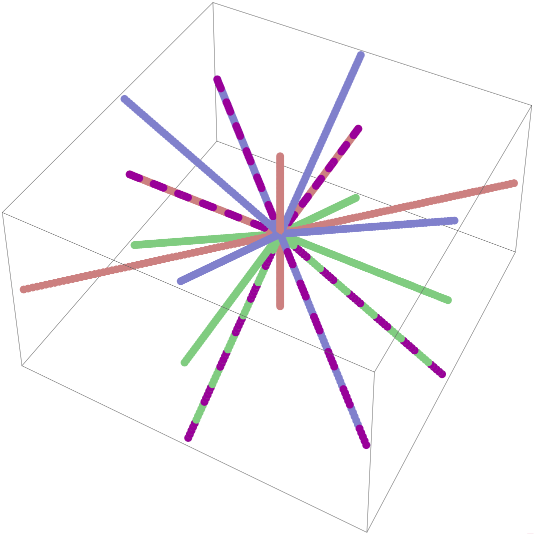
Example: Solving B_3 in char. 2



Example: Solving B_3 in char. 2



Example: Solving B_3 in char. 2



Intermezzo: Derivation Algebra

L a Lie algebra,

Definition (Derivation Algebra)

$$\mathrm{Der}(L) = \{D \in \mathrm{End}(L) \mid D[x, y] = [Dx, y] + [x, Dy]\}.$$

Observations:

- $\mathrm{Der}(L)$ with $[D, E] = DE$ is a Lie algebra:
- $L \subseteq \mathrm{Der}(L)$ via ad:

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$$\begin{aligned} [D, [E, F]](x) &= D(EFx) = D([E, F(x)]) \\ &= [DE, F(x)] + [E, DF(x)] \\ &= [[D, E], F](x) + [E, [D, F]](x) \\ &= (-[E, [F, D]] - [F, [D, E]])(x) \end{aligned}$$

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L a Lie algebra,

Definition (Derivation Algebra)

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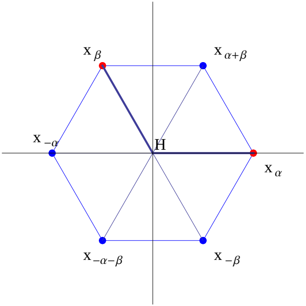
- $L \subseteq \mathrm{Der}(L)$ via ad:

$$\mathrm{ad}_t([x,y]) = [t,[x,y]] = [x,[t,y]] + [[t,x],y]$$

Example: Solving A_2 in char. 3



Type	Eigenspaces	Composition
Ad:	$0^2, 1^6$	$\frac{1}{7}$
SC:	$0^2, 3^2$	$\frac{7}{1}$

Observations:

- There is only one "7",
- $\mathrm{Der}(L^{\mathrm{sc}}) = L^{\mathrm{ad}}$.

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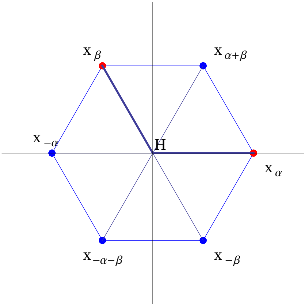
Example: Solving A_2 in char. 3



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Conclusion and Future Research

- Main challenges for computing Chevalley bases in small characteristic:
  - Multidimensional eigenspaces,
  - Broken root chains,
- Found solutions for majority of the cases,
  - And implemented these in MAGMA,
- Bigger picture:
  - Recognition of groups or Lie algebras,
  - Finding conjugators for Lie group elements,
  - Finding automorphisms of Lie algebras,
  - ...

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The End!

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