Construction of Chevalley Bases of Lie Algebras

Dan Roozemond
Joint work with Arjeh M. Cohen

October 29, 2008, Symbolic Computation Seminar,
NC State University, Department of Mathematics

http://www.win.tue.nl/∼droozemo/ (or Google)

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Outline

1. Why study Lie algebras?
2. Defining Lie algebras
   - Root system
   - Root datum
   - Lie algebra
3. Examples
4. Computing Chevalley Bases
   - Why?
   - How?
   - Strange things in small characteristic
   - Solving these things
5. Conclusion, Future research

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Why Study Lie Algebras?

- Study groups by Lie algebras:
  - Simple algebraic group $G \leftrightarrow$ Unique Lie algebra L
  - Many properties carry over to L
  - Easier to calculate in L
  - $G \leq \mathrm{Aut}(L)$, often even $G = \mathrm{Aut}(L)$
- Opportunities for:
  - Recognition
  - Conjugation
  - ...
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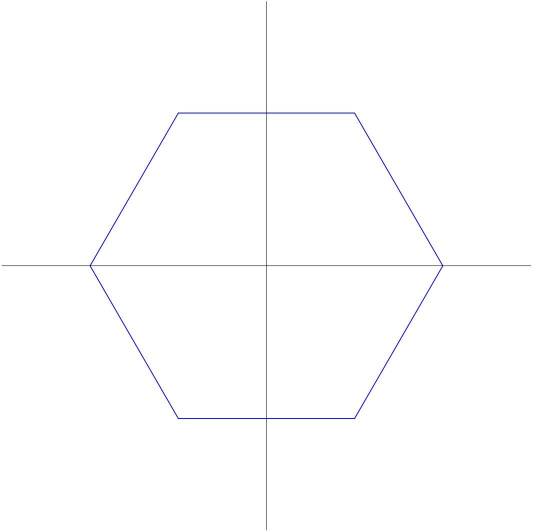
Root Systems



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- A root system of type A_2
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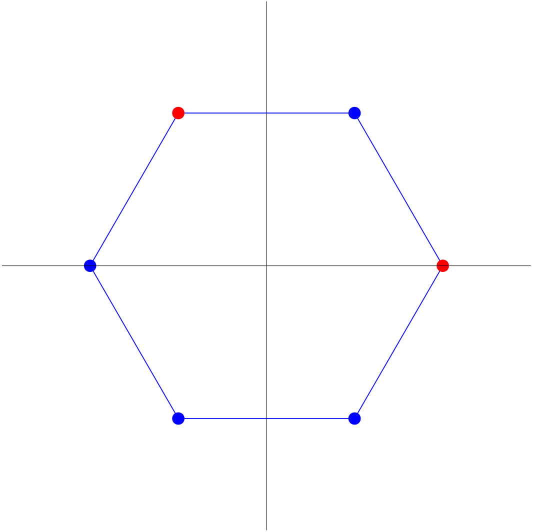

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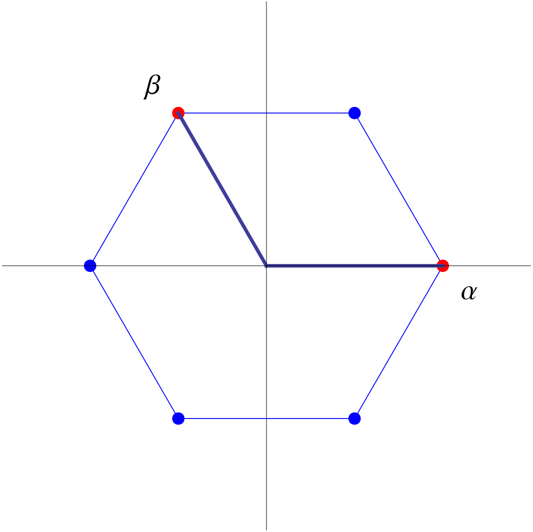

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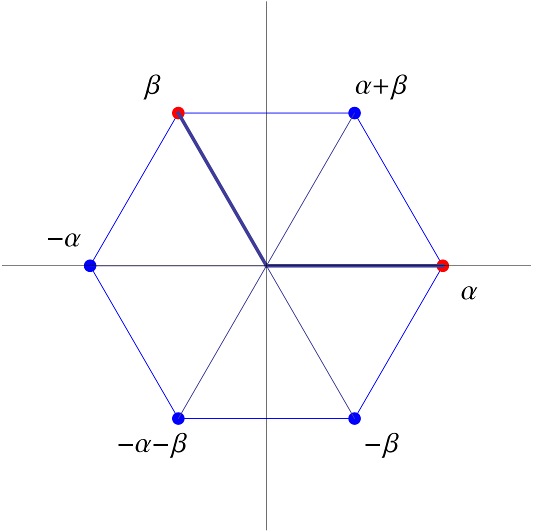

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Definition (Root Datum)

$$R = (X, \Phi, Y, \Phi^\vee), \quad \langle\cdot,\cdot\rangle : X \times Y \to \mathbb{Z},$$

- X, Y: dual free $\mathbb{Z}$-modules,
- put in duality by $\langle\cdot,\cdot\rangle$,
- $\Phi \subseteq X$: roots,
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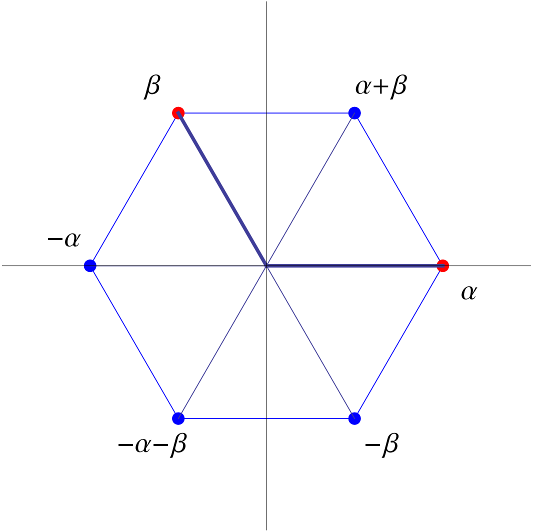

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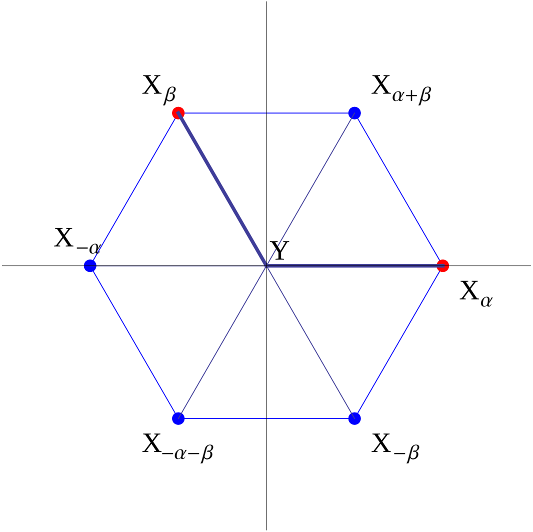

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Formal basis : $L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha$,

Multiplication : $[\cdot, \cdot]$

with bilinear antisymmetric multiplication defined by

- $y, z \in Y$: $[y, z] = 0$,
- $y \in Y, \beta \in \Phi$: $[X_\beta, y] = \langle \beta, y \rangle X_\beta$,
- $\alpha \in \Phi$: $[X_{-\alpha}, X_\alpha] = \alpha^\vee$,
- $\alpha, \beta \in \Phi$: $[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$
- $+$ Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Such a basis: a *Chevalley basis*.

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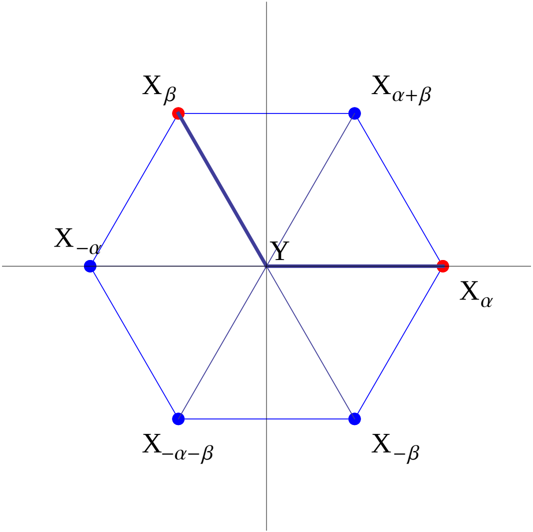
Root Systems → Lie Algebras



Chevalley Bases

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$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

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$\mathfrak{sl}_2$: Trace 0 matrices.

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[a, b] := ab - ba$$

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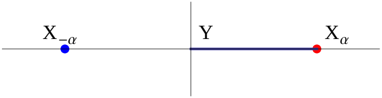

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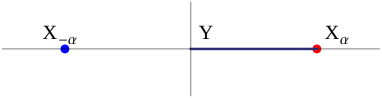

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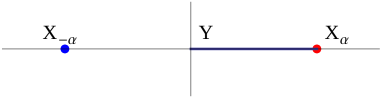

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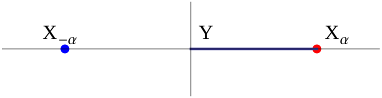

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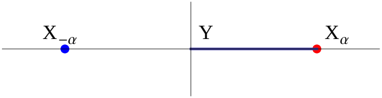

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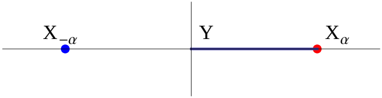

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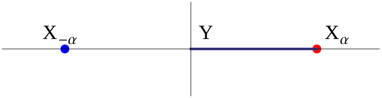

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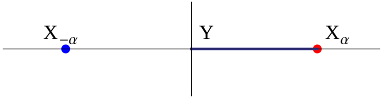

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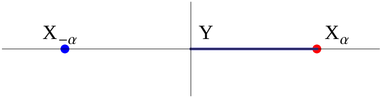

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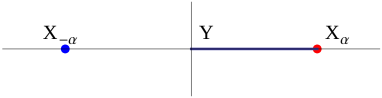

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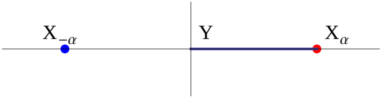

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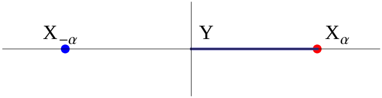

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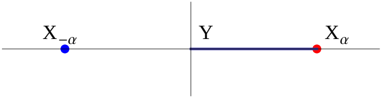

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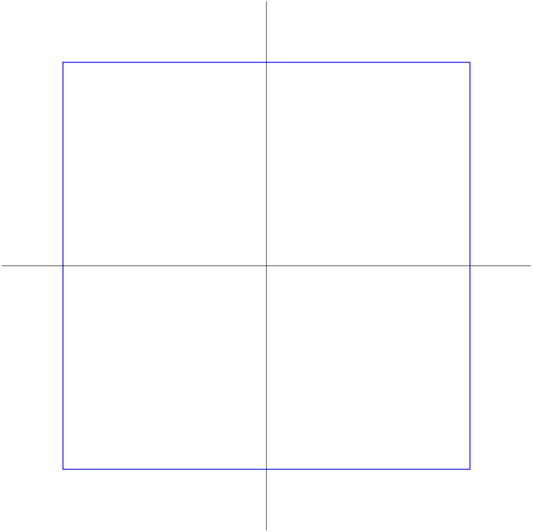

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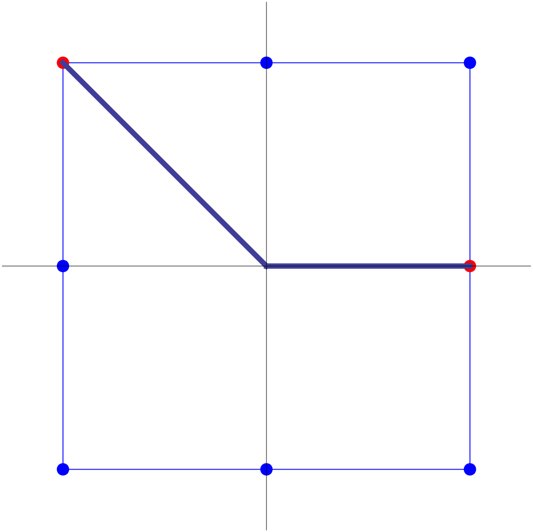

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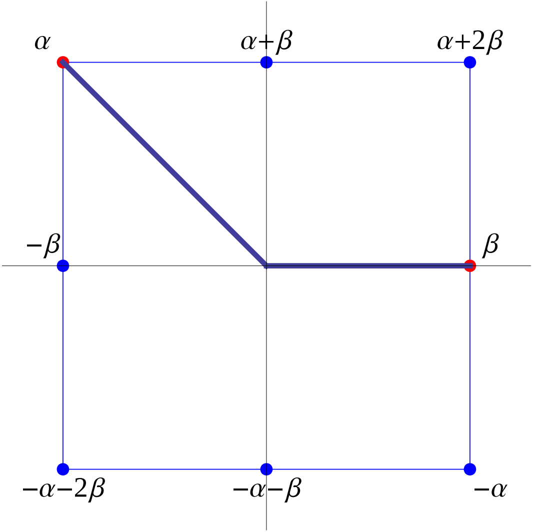

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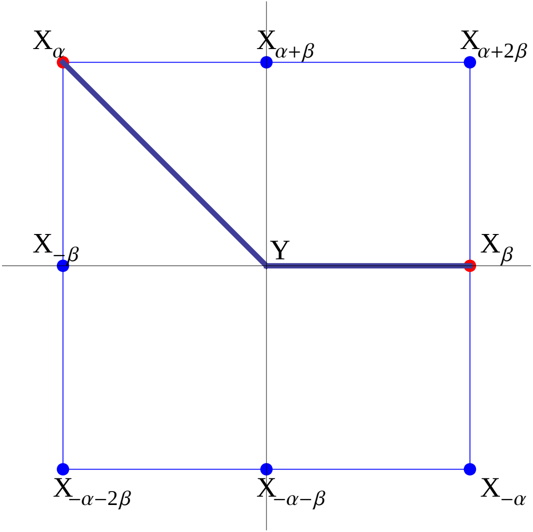
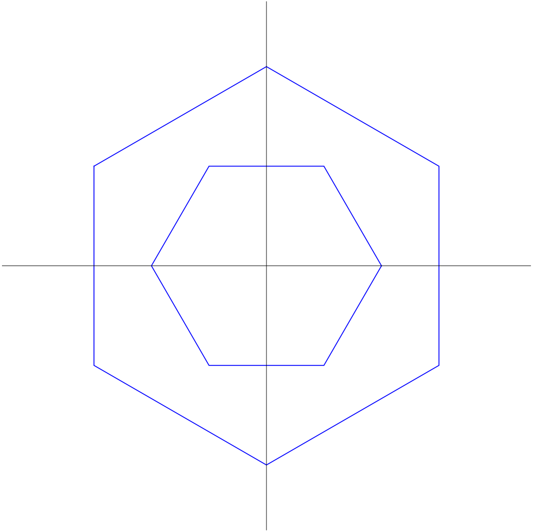


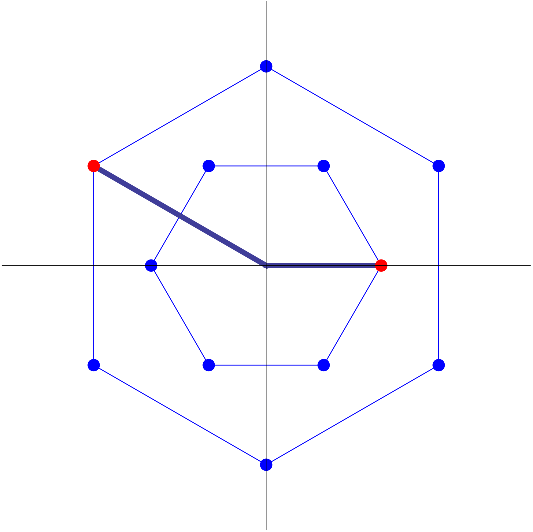
- A square
- A root system of type B_2
- A Lie algebra of type B_2

Example: G_2



- Two hexagons
- A root system of type G_2
- A Lie algebra of type G_2

Example: G_2



- Two hexagons
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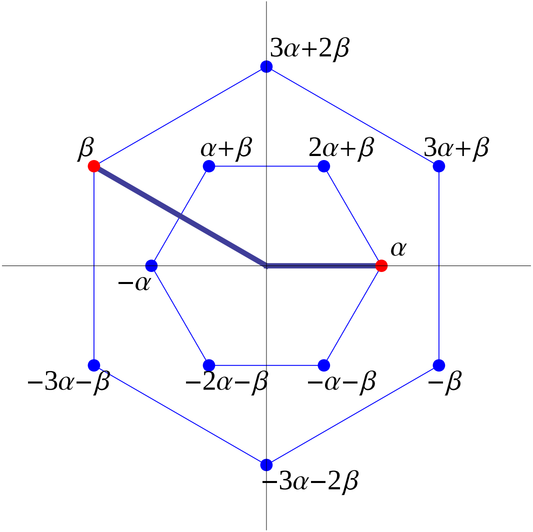
Example: G_2



- Two hexagons
- A root system of type G_2
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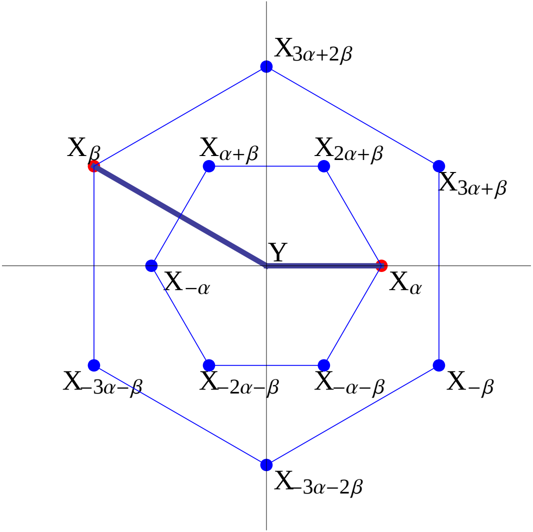
Example: G_2



- Two hexagons
- A root system of type G_2
- A Lie algebra of type G_2

Outline

1. Why study Lie algebras?
2. Defining Lie algebras
   - Root system
   - Root datum
   - Lie algebra
3. Examples
4. Computing Chevalley Bases
   - Why?
   - How?
   - Strange things in small characteristic
   - Solving these things
5. Conclusion, Future research

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Chevalley Bases

- Recap:

Definition (Chevalley Lie Algebra)

$$\begin{array}{ll} \text{Formal basis}: & L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}X_\alpha\,, \\ \text{Multiplication}: & [\cdot,\cdot] \end{array}$$

$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

- Idea: Given any Lie algebra, find a Chevalley basis.
- Why?
- Because transformation between two Chevalley bases is automorphism of L!

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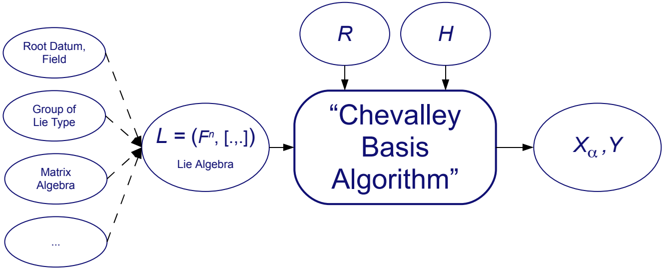
The Game



(Cohen/Murray, indep. Ryba)

Also given: Root datum R, splitting Cartan subalgebra $H = Y \otimes \mathbb{F}$

(De Graaf, Murray)

Char. 0, p $\geq$ 5: Implemented in GAP, Magma

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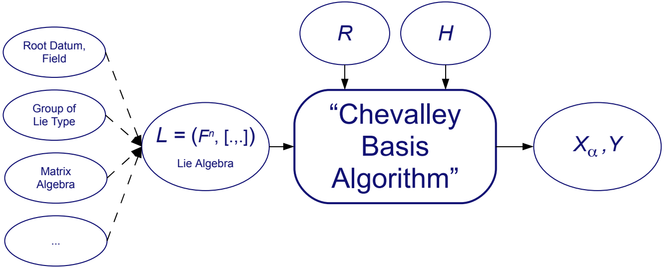
The Game



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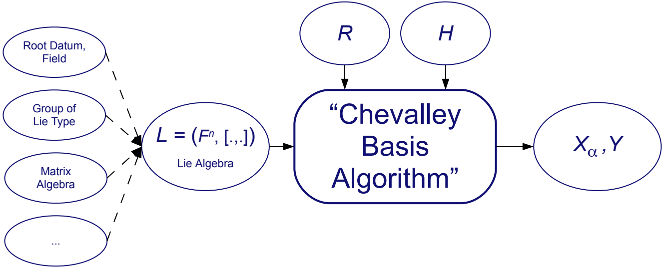
The Game



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Chevalley Basis Algorithm

CHEVALLEYBASIS

in: A simple Lie algebra L,
a splitting Cartan subalgebra H of L, and
a root datum $R = (X, \Phi, Y, \Phi^\vee)$.

out: A Chevalley basis B for L with respect to H and R.

begin

let $\{E_1, \ldots, E_m\} = \text{DIAGONALIZE}(L, H)$,

let $\{\overline{X}_1, \ldots, \overline{X}_{|\Phi|}\} = \text{STRAIGHTEN}(L, \{E_1, \ldots, E_m\})$,

let $\imath = \text{IDENTIFYROOTS}(L, R, \{\overline{X}_1, \ldots, \overline{X}_{|\Phi|}\})$,

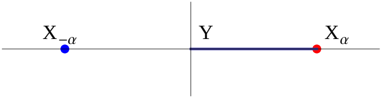
let $[X_\alpha \mid \alpha \in \Phi], [h_1, \ldots, h_{\text{rnk}(\Phi)}] = \text{SCALETOBASIS}(L, H, \{\overline{X}_1, \ldots, \overline{X}_{|\Phi|}\}, \imath)$,

return $[X_\alpha \mid \alpha \in \Phi], [h_1, \ldots, h_{\text{rnk}(\Phi)}]$.

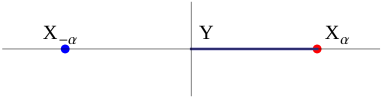
end

Algorithm: Finding a Chevalley Basis

Strange things in small characteristic (I)



	X_α	$X_{-\alpha}$	y
X_α	0	$-y$	$2X_\alpha$
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$
y	$-2X_\alpha$	$2X_{-\alpha}$	0

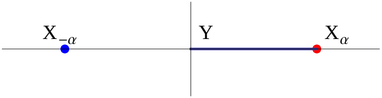


	X_α	$X_{-\alpha}$	y
X_α	0	$-2y$	X_α
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$
y	$-X_\alpha$	$X_{-\alpha}$	0

Observe:

- $y \mapsto \frac{1}{2}y$ maps $\mathrm{Lie}({A_1}^{sc}, \mathbb{F})$ to $\mathrm{Lie}({A_1}^{ad}, \mathbb{F})$,
- So $\mathrm{Lie}({A_1}^{sc}, \mathbb{F}) \cong \mathrm{Lie}({A_1}^{ad}, \mathbb{F})$,
- Except if $\mathrm{char}(\mathbb{F}) = 2$!

Strange things in small characteristic (I)



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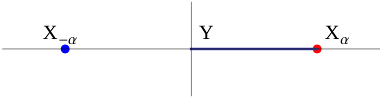
y	$-2X_\alpha$	$2X_{-\alpha}$	0



	X_α	$X_{-\alpha}$	y
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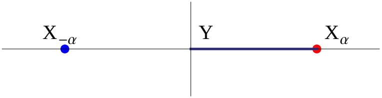

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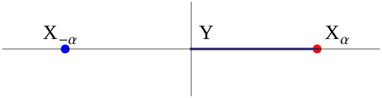

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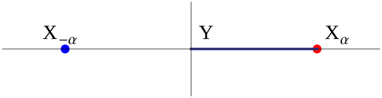
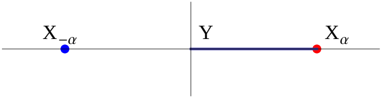
Strange things in small characteristic (II)



	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-y$	$2X_\alpha$	(2)
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)



	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-2y$	X_α	(1)
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
y	$-X_\alpha$	$X_{-\alpha}$	0	(0)

- Use action of Y to diagonalize L and find X_α,
- Except if the characteristic is 2!

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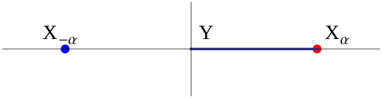
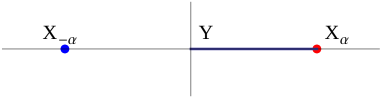
Strange things in small characteristic (II)





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$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)

	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-2y$	X_α	(1)
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
y	$-X_\alpha$	$X_{-\alpha}$	0	(0)

- Use action of Y to diagonalize L and find X_α,
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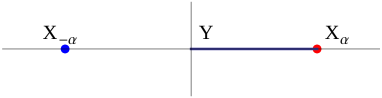
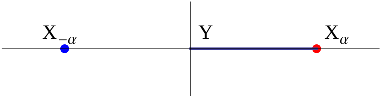
Strange things in small characteristic (II)



	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-y$	$2X_\alpha$	(2)
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)



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$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
y	$-X_\alpha$	$X_{-\alpha}$	0	(0)

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Strange things in small characteristic (III)

	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$\vdots$				

Strange things in small characteristic (III)

	$\ldots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2, -1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3, 2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1, 1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1, 0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3, -1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0, 1)$
$\vdots$				

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Strange things in small characteristic (III)

	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$
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$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$
$\vdots$				

Strange things in small characteristic (III)

	…	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
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$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$
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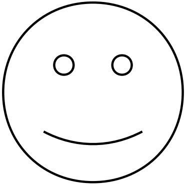

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Strange things in small characteristic (III)

	$\ldots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$GF(3^m)$	
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$	$(-1,-1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$	$(0,-1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
$\vdots$						

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	...	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	GF(3^m)	
X_α		$2X_\alpha$	$-X_\alpha$	$(2, -1)$	$(-1, -1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3, 2)$	$(0, -1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1, 1)$	$(-1, 1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1, 0)$	$(1, 0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3, -1)$	$(0, -1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0, 1)$	$(0, 1)$	($!^2$)
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2, 1)$	$(1, 1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3, -2)$	$(0, 1)$	($!^2$)
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1, -1)$	$(1, -1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1, 0)$	$(-1, 0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3, 1)$	$(0, 1)$	($!^2$)
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0, -1)$	$(0, -1)$	(!)
$\vdots$						

	$\ldots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$\mathrm{GF}(3^m)$	
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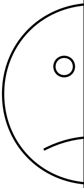
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	$(!^2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	$(!^2)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
$\vdots$						



Multidimensional Eigenspaces

Char.	Root datum	Eigenspace dims
3	$A_2{}^{sc}$	3^2
3	G_2	$1^6, 3^2$
2	$A_3{}^{sc}, A_3^{(a)\star}$	4^3
2	$B_n{}^{ad}$ $(n \geq 2)$	$2^n, 4^{\binom{n}{2}}$
2	$B_2{}^{sc}$	4^2
2	$B_3{}^{sc}$	6^3
2	$B_4{}^{sc}$	$2^4, 8^3$
2	$B_n{}^{sc}$ $(n \geq 5)$	$2^n, 4^{\binom{n}{2}}$
2	$C_n{}^{ad}$ $(n \geq 3)$	$2n^1, 2^{2\binom{n}{2}}$
2	$C_n{}^{sc}$ $(n \geq 3)$	$2n^1, 4^{\binom{n}{2}}$
2	$D_4^{(a),(b),(a+b)\star}$	4^6
2	$D_4{}^{sc}$	8^3
2	$D_n^{(a)\star}, D_n{}^{sc}$ $(n \geq 5)$	$4^{\binom{n}{2}}$
2	F_4	$2^{12}, 8^3$
2	G_2	4^3
2	all remaining cases	$2^N (N = \lvert\Phi^+\rvert)$

Steinberg, 1961

Complete list of multiplicities of roots, for root data of adjoint type

Cohen, R., 2008

Complete list of multiplicities of roots, for all root data

Multidimensional Eigenspaces

Char.	Root datum	Eigenspace dims
3	$A_2{}^{sc}$	3^2
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2	$A_3{}^{sc}, A_3^{(a)\star}$	4^3
2	$B_n{}^{ad}$ $(n \geq 2)$	$2^n, 4^{\binom{n}{2}}$
2	$B_2{}^{sc}$	4^2
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Outline

1. Why study Lie algebras?
2. Defining Lie algebras
   - Root system
   - Root datum
   - Lie algebra
3. Examples
4. Computing Chevalley Bases
   - Why?
   - How?
   - Strange things in small characteristic
   - Solving these things
5. Conclusion, Future research

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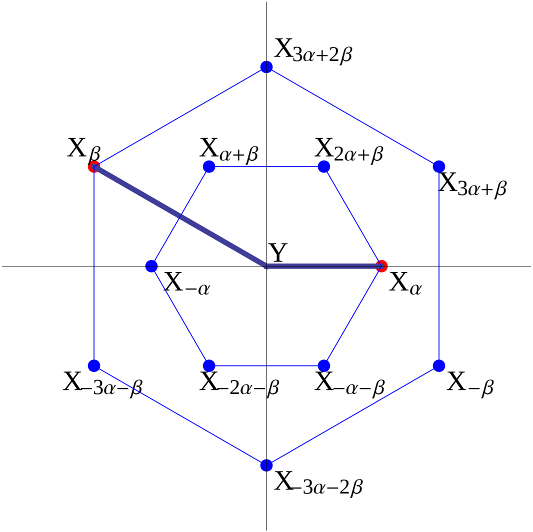
Multidimensional Eigenspaces

General Solution Strategies:

1. Nullspaces (ex: G_2, char. 3),
2. Ideals (ex: B_3, char. 2),
3. Derivation Algebra (ex: A_2, char. 3)

Example: Solving G_2 in char. 3



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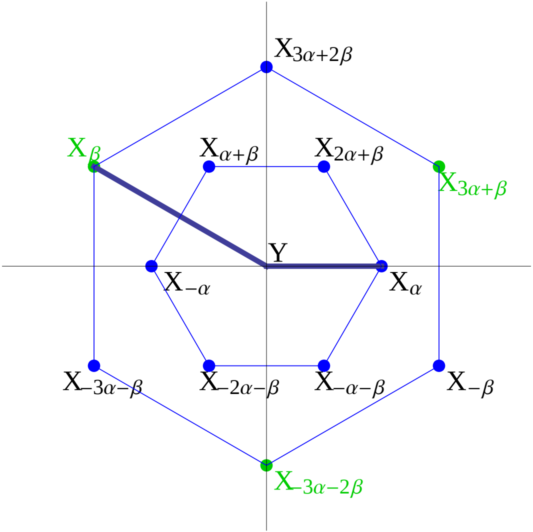

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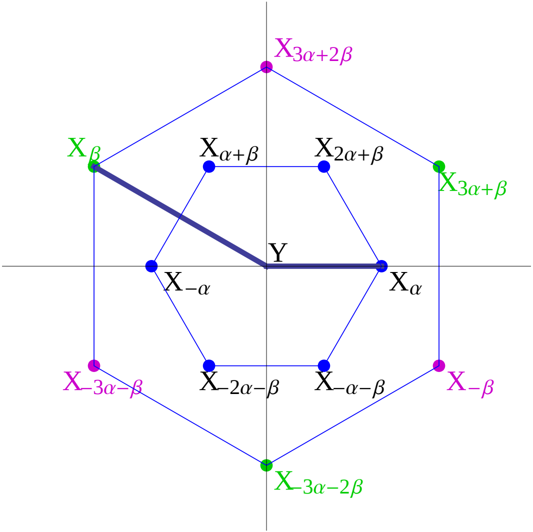

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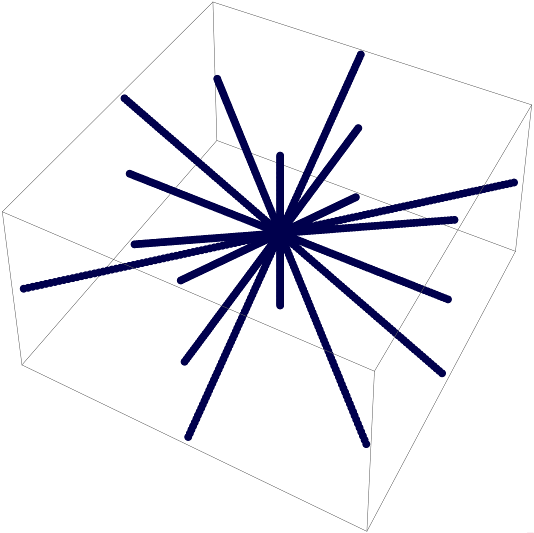
Example: Solving B_3 in char. 2



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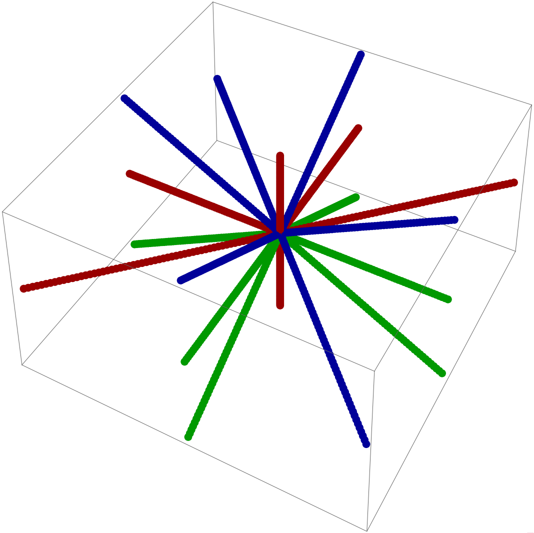

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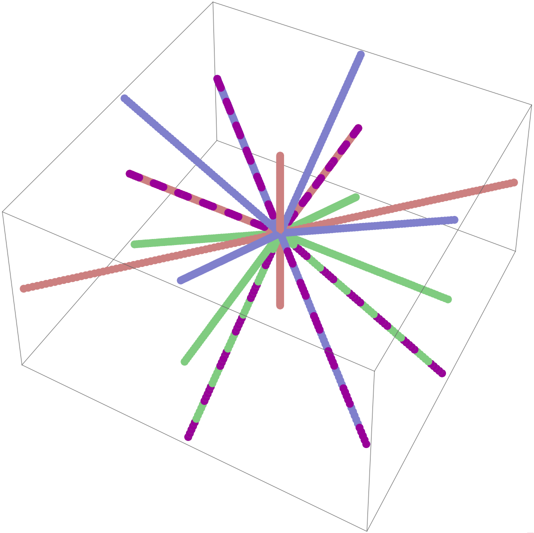

Intermezzo: Derivation Algebra

L a Lie algebra,

Definition (Derivation Algebra)

$$\mathrm{Der}(L) = \{D \in \mathrm{End}(L) \mid D[x, y] = [Dx, y] + [x, Dy]\}.$$

Observations:

- $\mathrm{Der}(L)$ with $[D, E] = DE$ is a Lie algebra:
- $L \subseteq \mathrm{Der}(L)$ via ad:

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$$\begin{aligned}
[D,[E,F]](x) &= D(EFx) = D([E,F(x)]) \\
&= [DE,F(x)] + [E,DF(x)] \\
&= [[D,E],F](x) + [E,[D,F]](x) \\
&= (-[E,[F,D]] - [F,[D,E]])(x)
\end{aligned}$$

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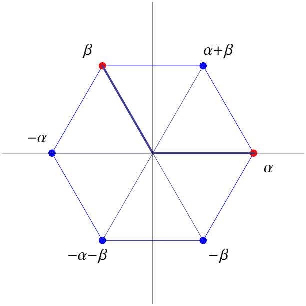
$$\mathrm{ad}_t([x, y]) = [t, [x, y]] = [x, [t, y]] + [[t, x], y]$$

Example: Solving A_2 in char. 3



Type	Eigenspaces	Composition
Ad:	$0^2, 1^6$	$\frac{1}{7}$
SC:	$0^2, 3^2$	$\frac{7}{1}$

Observations:

- There is only one "7",
- $\mathrm{Der}(L^{sc}) = L^{ad}$.

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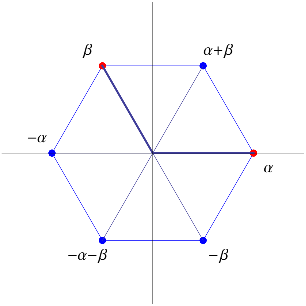
Example: Solving A_2 in char. 3



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Some Small Characteristic Solutions

Solving many small puzzles,

- For multidimensional eigenspaces:
  - Use nullspaces,
  - Intersect with nontrivial ideals (MeatAxe),
  - Consider Der(L),
- For $N_{\alpha,\beta} \equiv 0$:
  - Pinpoint scalar multiples of roots in a smart order,
- General methods:
  - Find root chains in $[\cdot,\cdot]$ instead of eigentuples,
  - Brute force approaches may be feasible in small cases,
  - Use ideals to reduce to smaller cases.

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Conclusion and Future Research

- Main challenges for computing Chevalley bases in small characteristic:
  - Multidimensional eigenspaces,
  - Broken root chains,
- Found solutions for majority of the cases,
  - And implemented these in MAGMA,
- Bigger picture:
  - Recognition of groups or Lie algebras,
  - Finding conjugators for Lie group elements,
  - Finding automorphisms of Lie algebras,
  - ...

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The End!

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