Lie Algebras over Fields of Small Characteristic

Dan Roozemond
Joint work with Arjeh Cohen

Applications of Computer Algebra
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TU/e Technische Universiteit Eindhoven University of Technology

Outline

1. Motivation
2. Root Data and Lie Algebras
3. Some Small Characteristic Trouble
4. Some Small Characteristic Solutions
5. Conclusions and Future Research

Why Lie Algebras?

- Study groups by Lie algebras:
  - Simple algebraic group $G \leftrightarrow$ Unique Lie algebra L
  - Many properties carry over to L
  - Easier to calculate in L
  - $G \leq \mathrm{Aut}(L)$, often even $G = \mathrm{Aut}(L)$
- Opportunities for:
  - Recognition
  - Conjugation
  - ...
- Important tool for Lie algebras: Chevalley basis
- This talk: *Lie algebras over fields of small characteristic:*
  - What special cases occur?
  - How to compute Chevalley bases?

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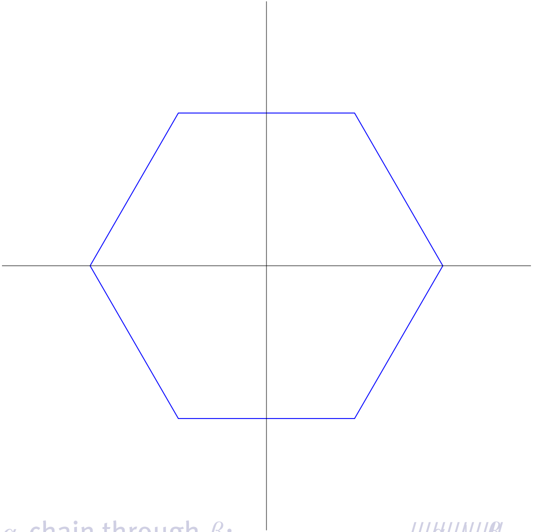

- A hexagon
- A root system of type A_2
- A Lie algebra of type A_2

α-chain through β: ~~$\beta - \alpha$~~, β, $\alpha + \beta$, ~~$2\alpha + \beta$~~ $\Rightarrow k = 0$

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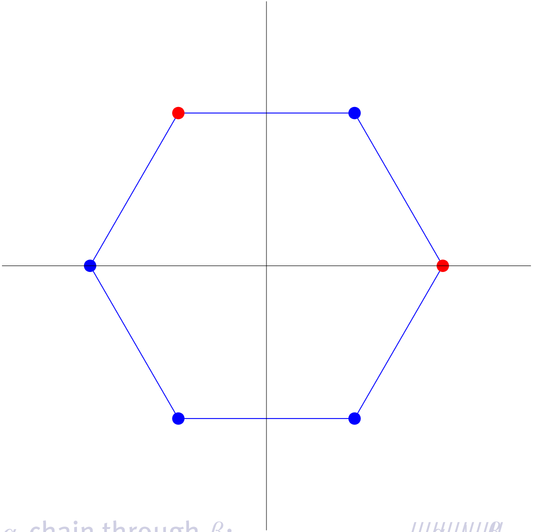
Root Systems



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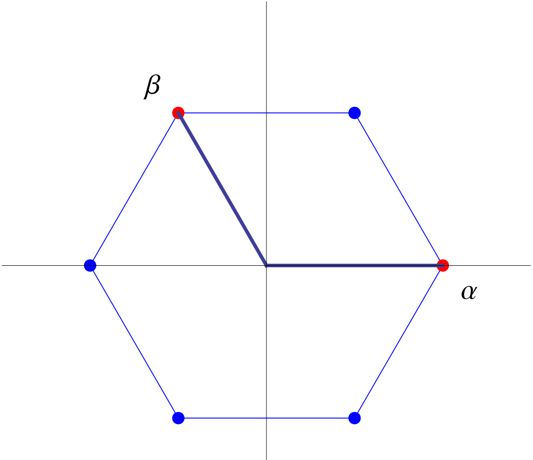

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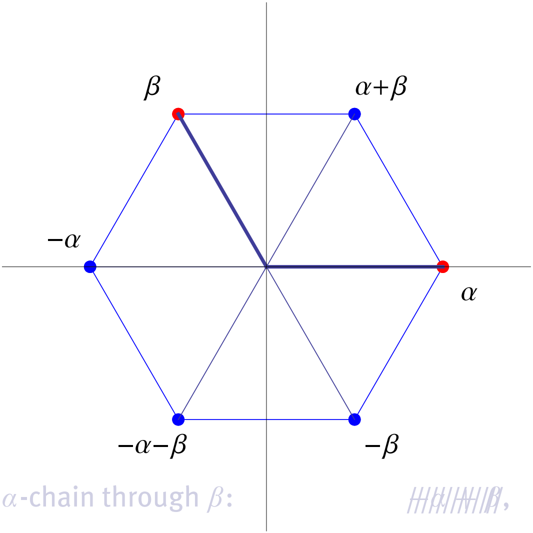

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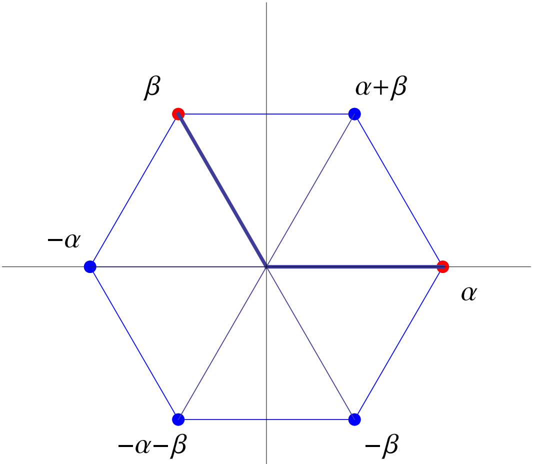

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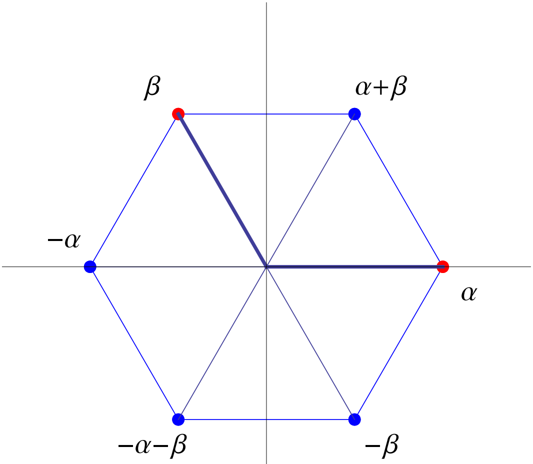

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Root Data

Definition (Root Datum)

$$R = (X, \Phi, Y, \Phi^\vee), \quad \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z},$$

- X, Y: dual free $\mathbb{Z}$-modules,
- put in duality by $\langle \cdot, \cdot \rangle$,
- $\Phi \subseteq X$: roots,
- $\Phi^\vee \subseteq Y$: coroots.

A root system Φ induces several non-isomorphic root data, the "extremes" known as *adjoint* and *simply connected*.

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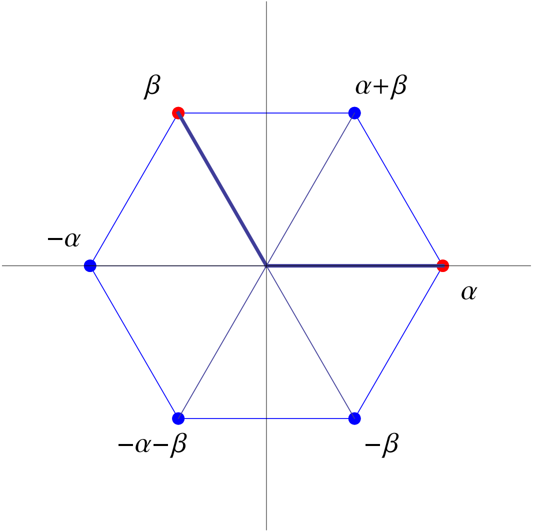

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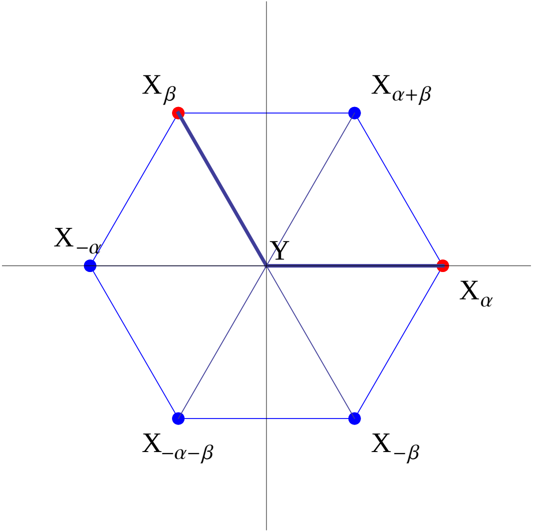

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Definition (Chevalley Lie Algebra)

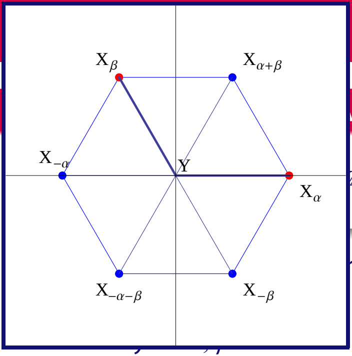
$$L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha ,$$

with bilinear antisymmetric multiplication defined by

- $y, z \in Y$: $[y, z] = 0$,
- $y \in Y, \beta \in \Phi$: $[X_\beta, y] = \langle \beta, y \rangle X_\beta$,
- $\alpha \in \Phi$: $[X_{-\alpha}, X_\alpha] = \alpha^\vee$,
- $\alpha, \beta \in \Phi$: $[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$
- $+$ some extra conditions.

Such a basis: a *Chevalley basis*.

Well known: $N_{\alpha,\beta}$'s can be chosen so that $N_{\alpha,\beta} = \pm(k+1)$



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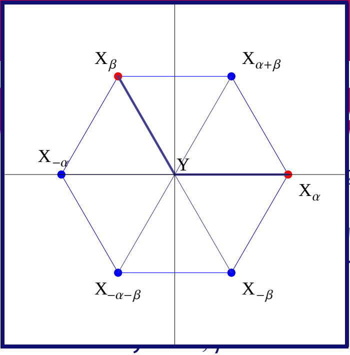

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$$L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_{\alpha},$$

$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

Why Chevalley bases? Because transformation between two Chevalley bases is automorphism of L!

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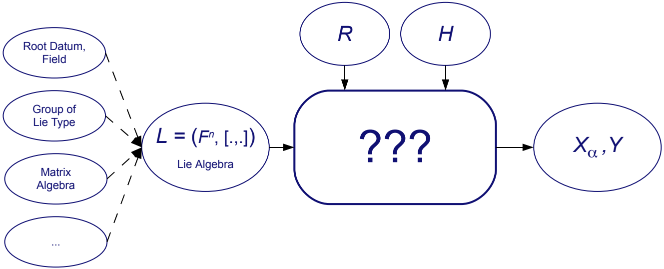
The Game

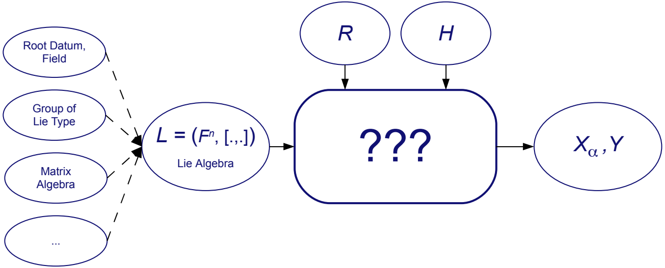


(Cohen/Murray, indep. Ryba)

Also given: Root datum R, splitting Cartan subalgebra $H = Y \otimes \mathbb{F}$

(De Graaf, Murray)

Char. 0, p $\geq$ 5: Implemented in GAP, Magma



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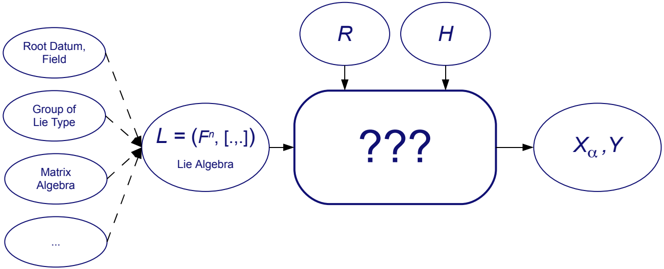

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Example: $\mathfrak{sl}_2$ / A_1

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
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$$[a, b] := ab - ba$$

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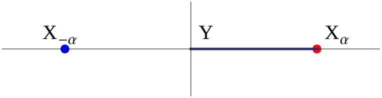

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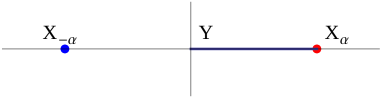

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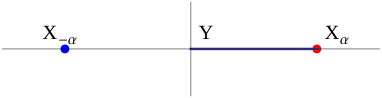

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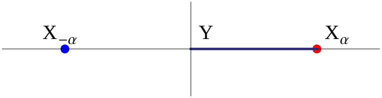

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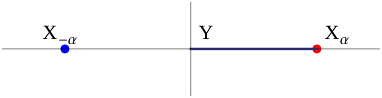

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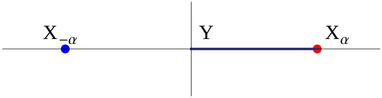
Example: $A_1{}^{sc}$ / $A_1{}^{ad}$



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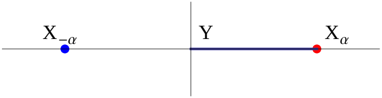

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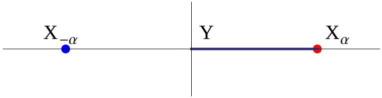

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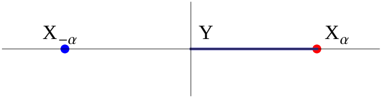

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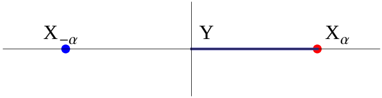

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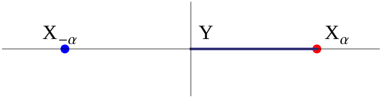
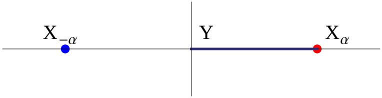


$$A_1{}^{ad}: X = Y = \mathbb{Z},$$
$$\Phi = \{\alpha = 1, -\alpha = -1\},$$
$$\Phi^\vee = \{\alpha^\vee = 2, -\alpha^\vee = -2\},$$

$$\begin{aligned} L_\mathbb{Z} &= Y \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}X_\alpha \\ &= \mathbb{Z}y \oplus \mathbb{Z}X_\alpha \oplus \mathbb{Z}X_{-\alpha}, \end{aligned}$$

	X_α	$X_{-\alpha}$	y
X_α	0	$-2y$	X_α
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$
y	$-X_\alpha$	$X_{-\alpha}$	0

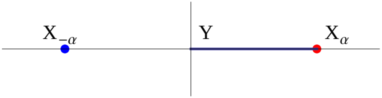
Example: $A_1{}^{sc}$ / $A_1{}^{ad}$



$$A_1{}^{sc}: X = Y = \mathbb{Z},$$
$$\Phi = \{\alpha = 2, -\alpha = -2\},$$
$$\Phi^\vee = \{\alpha^\vee = 1, -\alpha^\vee = -1\},$$

$$\begin{aligned} L_\mathbb{Z} &= Y \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}X_\alpha \\ &= \mathbb{Z}y \oplus \mathbb{Z}X_\alpha \oplus \mathbb{Z}X_{-\alpha}, \end{aligned}$$

	X_α	$X_{-\alpha}$	y
X_α	0	$-y$	$2X_\alpha$
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$
y	$-2X_\alpha$	$2X_{-\alpha}$	0



$$A_1{}^{ad}: X = Y = \mathbb{Z},$$
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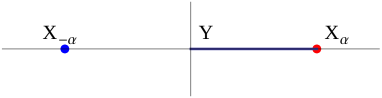
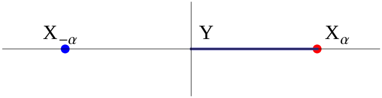
$$\begin{aligned} L_\mathbb{Z} &= Y \oplus \bigoplus_{\alpha\in\Phi} \mathbb{Z}X_\alpha \\ &= \mathbb{Z}y \oplus \mathbb{Z}X_\alpha \oplus \mathbb{Z}X_{-\alpha}, \end{aligned}$$

	X_α	$X_{-\alpha}$	y
X_α	0	$-2y$	X_α
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$
y	$-X_\alpha$	$X_{-\alpha}$	0

Example: $A_1{}^{sc}$ / $A_1{}^{ad}$





	X_α	$X_{-\alpha}$	y
X_α	0	$-y$	$2X_\alpha$
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$
y	$-2X_\alpha$	$2X_{-\alpha}$	0

	X_α	$X_{-\alpha}$	y
X_α	0	$-2y$	X_α
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$
y	$-X_\alpha$	$X_{-\alpha}$	0

Observe:

- $y \mapsto \frac{1}{2}y$ maps $\mathrm{Lie}(A_1{}^{sc}, \mathbb{F})$ to $\mathrm{Lie}(A_1{}^{ad}, \mathbb{F})$,
- So $\mathrm{Lie}(A_1{}^{sc}, \mathbb{F}) \cong \mathrm{Lie}(A_1{}^{ad}, \mathbb{F})$,
- Except if $\mathrm{char}(\mathbb{F}) = 2$!

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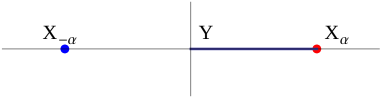
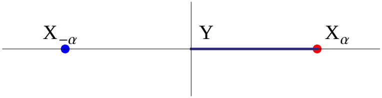
Example: $A_1{}^{sc}$ / $A_1{}^{ad}$



	X_α	$X_{-\alpha}$	y
X_α	0	$-y$	$2X_\alpha$
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$
y	$-2X_\alpha$	$2X_{-\alpha}$	0



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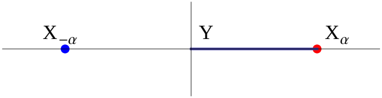
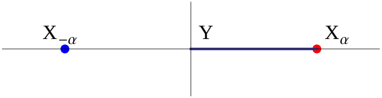
Example: A_1^{sc} / A_1^{ad}



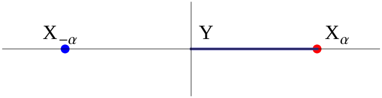
	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-y$	$2X_\alpha$	(2)
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)



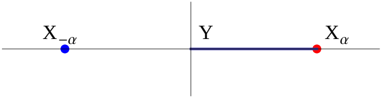
	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-2y$	X_α	(1)
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
y	$-X_\alpha$	$X_{-\alpha}$	0	(0)

- Use action of Y to diagonalize L and find X_α,
- Except if the characteristic is 2!

Example: $A_1{}^{sc}$ / $A_1{}^{ad}$



	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-y$	$2X_\alpha$	(2)
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)



	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-2y$	X_α	(1)
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
y	$-X_\alpha$	$X_{-\alpha}$	0	(0)

- Use action of Y to diagonalize L and find X_α,
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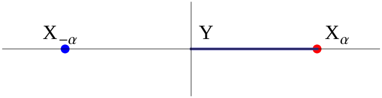
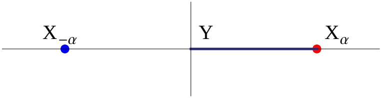


	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-y$	$2X_\alpha$	(2)
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)



	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-2y$	X_α	(1)
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
y	$-X_\alpha$	$X_{-\alpha}$	0	(0)

- Use action of Y to diagonalize L and find X_α,
- Except if the characteristic is 2!

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Example 2: G_2

	$\ldots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$\vdots$				

Example 2: G_2

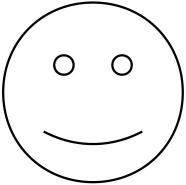
	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2, -1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3, 2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1, 1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1, 0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3, -1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0, 1)$
$\vdots$				

Example 2: G_2

	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$
$\vdots$				

Example 2: G_2

	$\ldots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2, -1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3, 2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1, 1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1, 0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3, -1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0, 1)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2, 1)$
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3, -2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1, -1)$
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1, 0)$
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3, 1)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0, -1)$
$\vdots$				



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Example 2: G_2

	...	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$GF(3^m)$	
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$	$(-1,-1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$	$(0,-1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
$\vdots$						

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Example 2: G_2

	…	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$GF(3^m)$	
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$	$(-1,-1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$	$(0,-1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	$(!^2)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	$(!^2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	$(!^2)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
$\vdots$						

Example 2: G_2

	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$GF(3^m)$	
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$	$(-1,-1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$	$(0,-1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	$(!^2)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	$(!^2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	$(!^2)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
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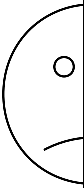

Outline

Some Small Characteristic Trouble

$$L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha,$$

- $y, z \in Y:$ $\quad [y, z] = 0,$
- $y \in Y, \beta \in \Phi:$ $\quad [X_\beta, y] = \langle \beta, y \rangle X_\beta,$
- $\alpha \in \Phi:$ $\quad [X_{-\alpha}, X_\alpha] = \alpha^\vee,$
- $\alpha, \beta \in \Phi:$ $\quad [X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$

Trouble:

1. Multidimensional eigenspaces,
2. Root chains are broken,
3. $k + 1 \equiv 0$, so $N_{\alpha,\beta} \equiv 0$,

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Multidimensional Eigenspaces

Steinberg, 1961

Complete list of multiplicities of roots, for root data of adjoint type

Cohen, R., 2008

Complete list of multiplicities of roots, for all root data

Char.	Root datum	Eigenspace dims
3	$A_2{}^{sc}$	3^2
3	G_2	$1^6, 3^2$
2	$A_3{}^{sc}, A_3^{(a)\star}$	4^3
2	$B_n{}^{ad}$ $(n \geq 2)$	$2^n, 4^{(n^2-n)/2}$
2	$B_2{}^{sc}$	4^2
2	$B_3{}^{sc}$	6^3
2	$B_4{}^{sc}$	$2^4, 8^3$
2	$B_n{}^{sc}$ $(n \geq 5)$	$2^n, 4^{(n^2-n)/2}$
2	$C_n{}^{ad}$ $(n \geq 3)$	$2n^1, 2^{n^2-n}$
2	$C_n{}^{sc}$ $(n \geq 3)$	$2n^1, 4^{(n^2-n)/2}$
2	$D_4^{(a),(b),(a+b)\star}$	4^6
2	$D_4{}^{sc}$	8^3
2	$D_n^{(a)\star}, D_n{}^{sc}$ $(n \geq 5)$	$4^{\binom{n}{2}}$
2	F_4	$2^{12}, 8^3$
2	G_2	4^3
2	all remaining cases	$2^N (N = \lvert\Phi^+\rvert)$

Multidimensional Eigenspaces

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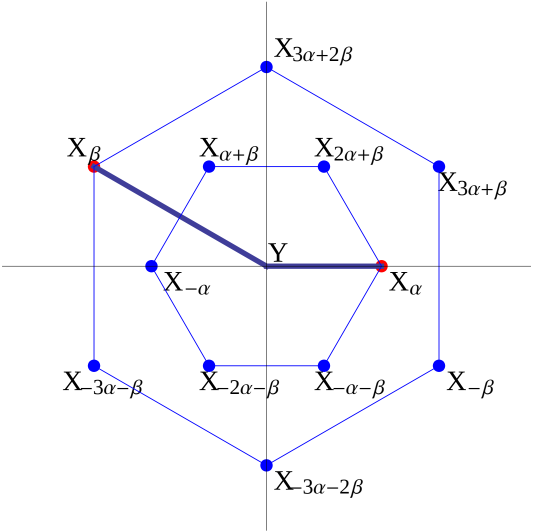
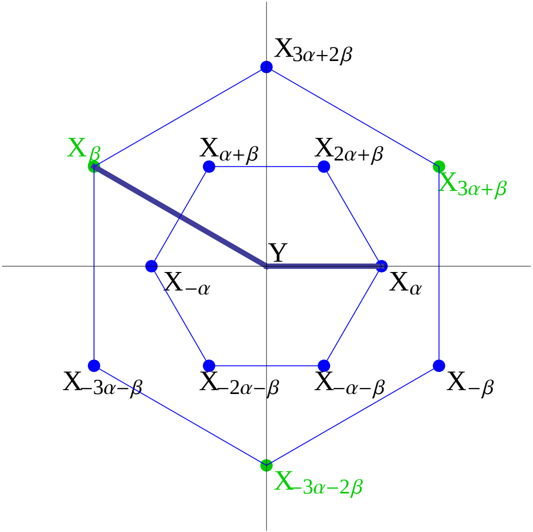
2	${B_3}^{sc}$	6^3
2	${B_4}^{sc}$	$2^4, 8^3$
2	${B_n}^{sc}$ $(n \geq 5)$	$2^n, 4^{(n^2-n)/2}$
2	${C_n}^{ad}$ $(n \geq 3)$	$2n^1, 2^{n^2-n}$
2	${C_n}^{sc}$ $(n \geq 3)$	$2n^1, 4^{(n^2-n)/2}$
2	$D_4^{(a),(b),(a+b)\star}$	4^6
2	${D_4}^{sc}$	8^3
2	$D_n^{(a)\star}, {D_n}^{sc}$ $(n \geq 5)$	$4^{\binom{n}{2}}$
2	F_4	$2^{12}, 8^3$
2	G_2	4^3
2	all remaining cases	$2^N (N = \lvert\Phi^+\rvert)$

Outline

Example: Solving G_2 in char. 3



Example: Solving G_2 in char. 3



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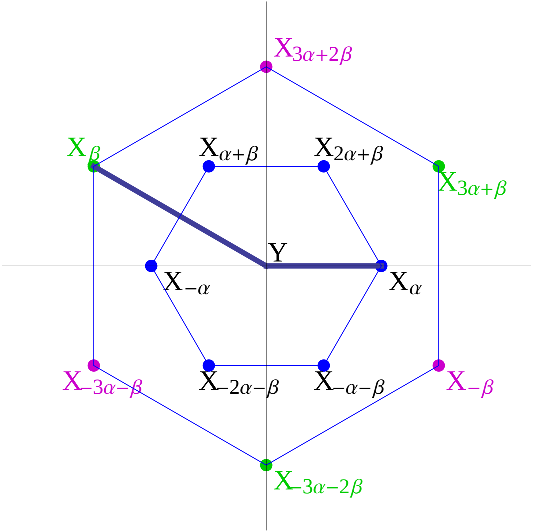

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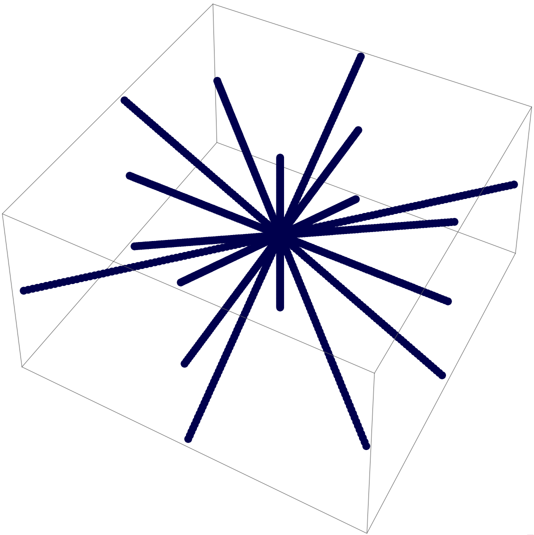
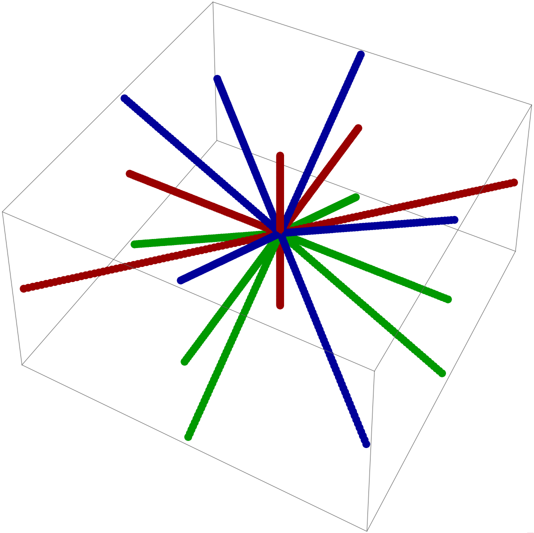
Example: Solving B_3 in char. 2



Example: Solving B_3 in char. 2



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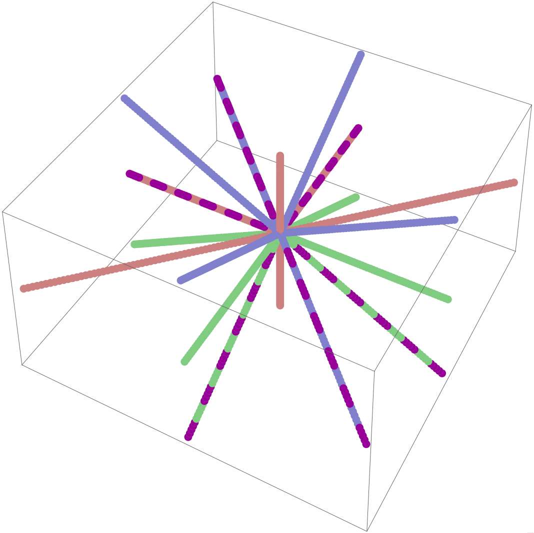

Some Small Characteristic Solutions

Solving many small puzzles,

- For $N_{\alpha,\beta} \equiv 0$:
  - Pinpoint scalar multiples of roots in a smart order,
- For multidimensional eigenspaces:
  - "Pivot" using eigenspaces with smaller dimension,
  - Intersect with nontrivial ideals (MeatAxe),
  - Consider $\mathrm{Der}(L)$,
- General methods:
  - Find root chains in $[\cdot, \cdot]$ instead of eigentuples,
  - Small cases may be brute forced,
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Conclusion and Future Research

- Main challenges for computing Chevalley bases in small characteristic:
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  - ...
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