Chevalley Bases of Lie Algebras over Fields of Small Characteristic

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Joint work with Arjeh Cohen

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/ department of mathematics and computer science

TU/e Technische Universiteit Eindhoven University of Technology

Outline

1. Root Data and Lie Algebras
   - Definition
   - A_1
   - G_2
2. Small Characteristic Trouble
   - Overview
   - Multidimensional Eigenspaces
3. Some Small Characteristic Solutions
4. Conclusions and Future Research

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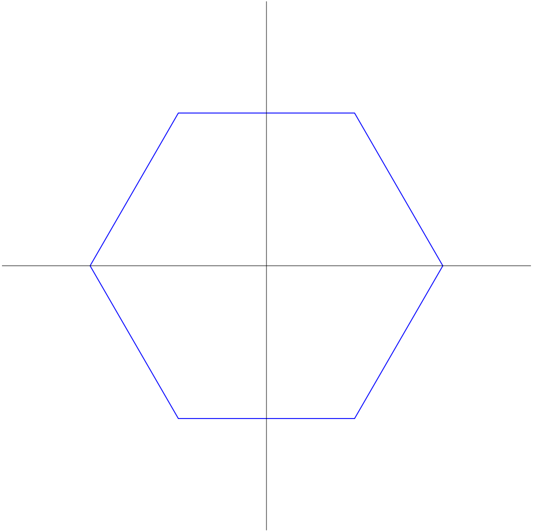
Root Systems



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- A root system of type A_2
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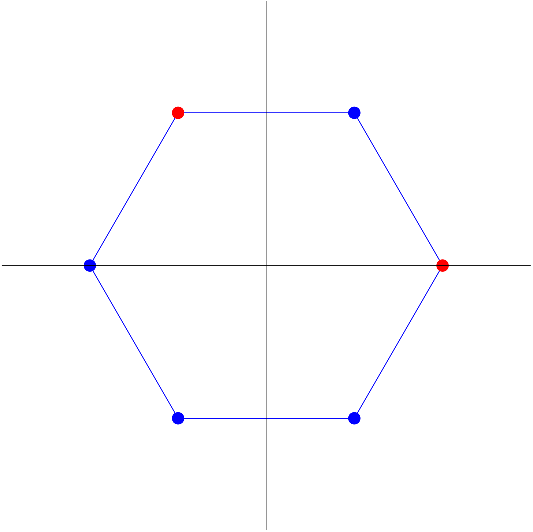

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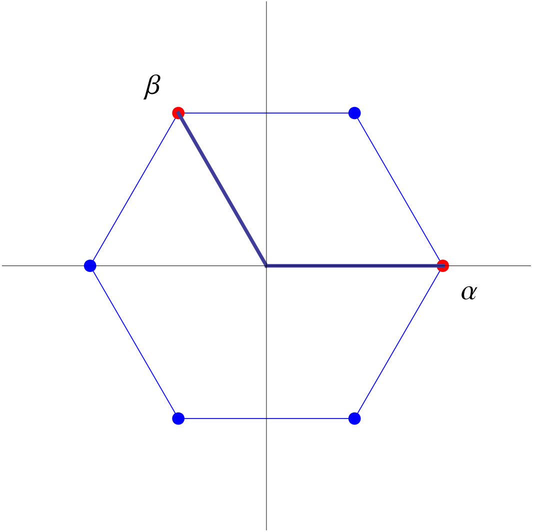

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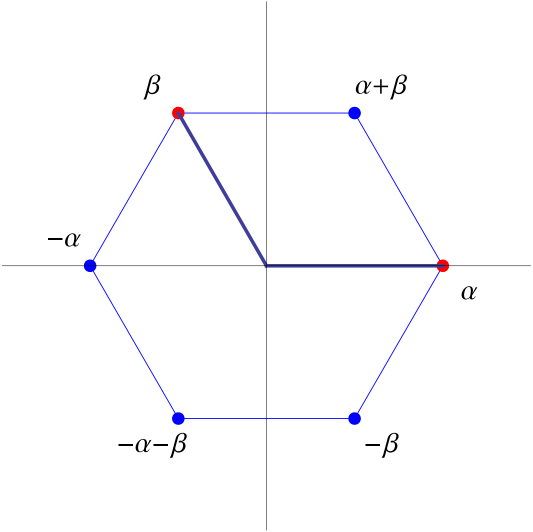

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Root Data

Definition (Root Datum)

$$R = (X, \Phi, Y, \Phi^\vee), \quad \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z},$$

- X, Y: dual free $\mathbb{Z}$-modules,
- put in duality by $\langle \cdot, \cdot \rangle$,
- $\Phi \subseteq X$: roots,
- $\Phi^\vee \subseteq Y$: coroots.

A root system Φ induces several non-isomorphic root data, the "extremes" known as *adjoint* and *simply connected*.

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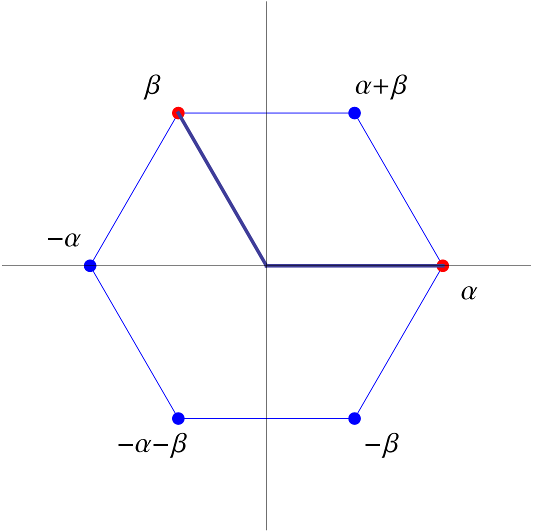

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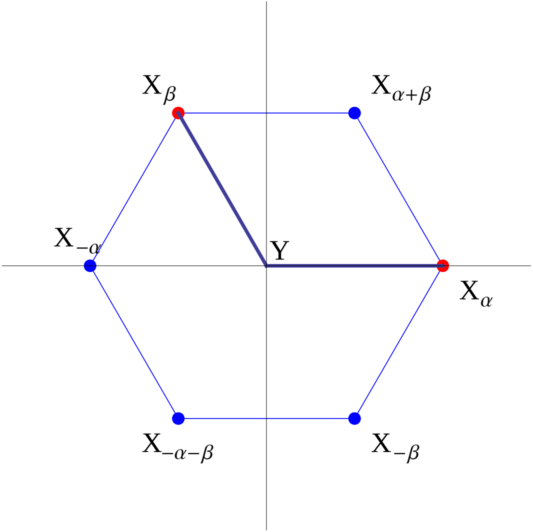

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Definition (Chevalley Lie Algebra)

$$L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha,$$

with bilinear antisymmetric multiplication defined by

- $y, z \in Y:\quad [y, z] = 0,$
- $y \in Y, \beta \in \Phi:\quad [X_\beta, y] = \langle \beta, y \rangle X_\beta,$
- $\alpha \in \Phi:\quad [X_{-\alpha}, X_\alpha] = \alpha^\vee,$
- $\alpha, \beta \in \Phi:\quad [X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$
- $+$ some extra conditions.

Such a basis: a *Chevalley basis*. Well known: $N_{\alpha,\beta}$ can be chosen so that $N_{\alpha,\beta} = \pm(k+1)$, where k maximal such that $\alpha - k\beta$ is a root.

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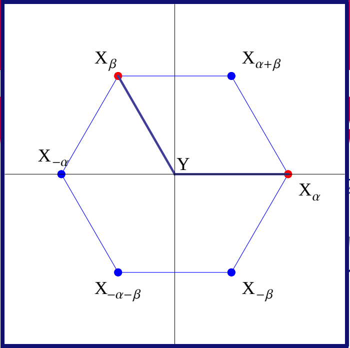

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$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

Finding Chevalley Bases:

Given: Lie algebra $L_{\mathbb{F}}$ with multiplication $[\cdot, \cdot]$,

Want: To find back (basis of) Y and X_α $(\alpha \in \Phi)$.

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Also given: Root datum R, splitting Cartan subalgebra $Y \otimes \mathbb{F}$

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Example: $\mathfrak{sl}_2$ / A_1

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
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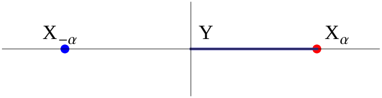

$$A_1{}^{\mathrm{sc}}: X = Y = \mathbb{Z},$$
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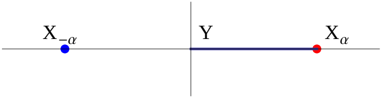

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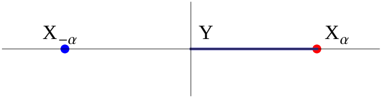

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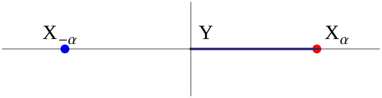

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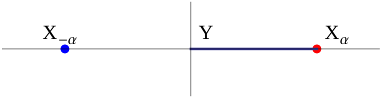

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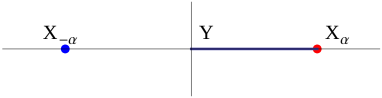

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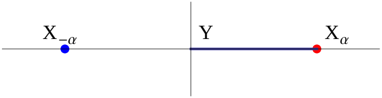

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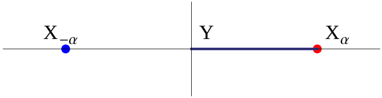

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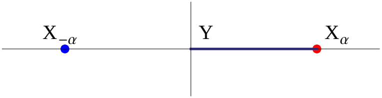

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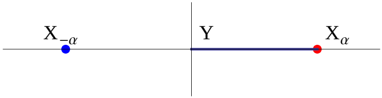

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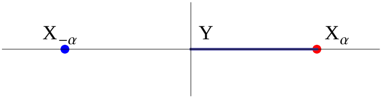

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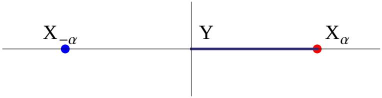

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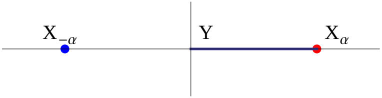

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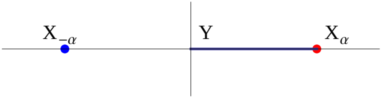

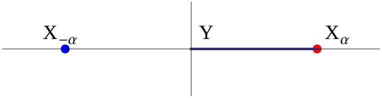

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Observe:

- $y \mapsto \frac{1}{2}y$ maps $A_1{}^{sc}$ to $A_1{}^{ad}$,
- Except if the characteristic is 2!

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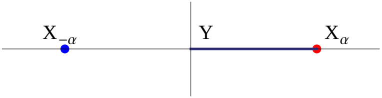

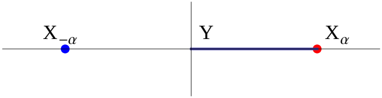

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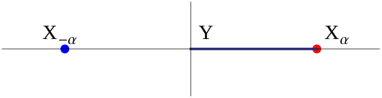
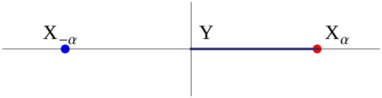
- Except if the characteristic is 2!

Example: $A_1{}^{sc}$ / $A_1{}^{ad}$



	X_α	$X_{-\alpha}$	y
X_α	0	$-y$	$2X_\alpha$
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$
y	$-2X_\alpha$	$2X_{-\alpha}$	0



	X_α	$X_{-\alpha}$	y
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$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$
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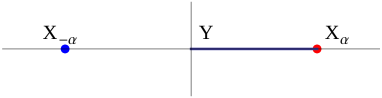
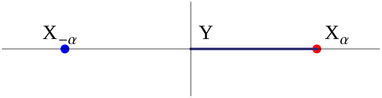


	X_α	$X_{-\alpha}$	y	$\mathbb{Z}^1$
X_α	0	$-y$	$2X_\alpha$	(2)
$X_{-\alpha}$	y	0	$-2X_{-\alpha}$	(-2)
y	$-2X_\alpha$	$2X_{-\alpha}$	0	(0)



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X_α	0	$-2y$	X_α	(1)
$X_{-\alpha}$	$2y$	0	$-X_{-\alpha}$	(-1)
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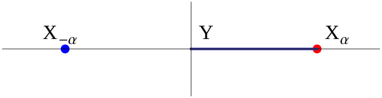

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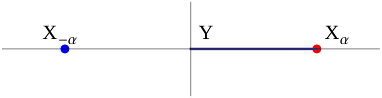

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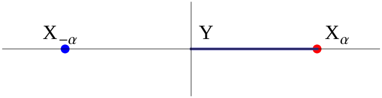

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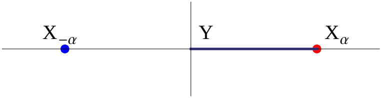

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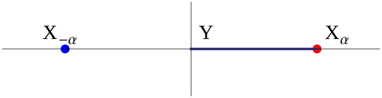

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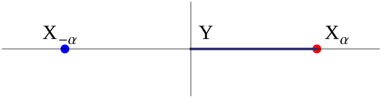

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Example 2: G_2

	$\ldots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$\vdots$				

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Example 2: G_2

	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$
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$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
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$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$
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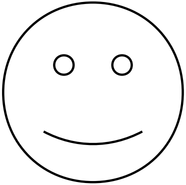

Example 2: G_2

	$\dots$	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$GF(3^m)$	
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$	$(-1,-1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$	$(0,-1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
$\vdots$						

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Example 2: G_2

	…	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	$GF(3^m)$	
X_α		$2X_\alpha$	$-X_\alpha$	$(2,-1)$	$(-1,-1)$	
X_β		$-3X_\beta$	$2X_\beta$	$(-3,2)$	$(0,-1)$	(!)
$X_{\alpha+\beta}$		$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	$(-1,1)$	$(-1,1)$	
$X_{2\alpha+\beta}$		$X_{2\alpha+\beta}$	0	$(1,0)$	$(1,0)$	
$X_{3\alpha+\beta}$		$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	$(3,-1)$	$(0,-1)$	(!)
$X_{3\alpha+2\beta}$		0	$X_{3\alpha+2\beta}$	$(0,1)$	$(0,1)$	$(!^2)$
$X_{-\alpha}$		$-2X_{-\alpha}$	$X_{-\alpha}$	$(-2,1)$	$(1,1)$	
$X_{-\beta}$		$3X_{-\beta}$	$-2X_{-\beta}$	$(3,-2)$	$(0,1)$	$(!^2)$
$X_{-\alpha-\beta}$		$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	$(1,-1)$	$(1,-1)$	
$X_{-2\alpha-\beta}$		$-X_{-2\alpha-\beta}$	0	$(-1,0)$	$(-1,0)$	
$X_{-3\alpha-\beta}$		$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	$(-3,1)$	$(0,1)$	$(!^2)$
$X_{-3\alpha-2\beta}$		0	$-X_{-3\alpha-2\beta}$	$(0,-1)$	$(0,-1)$	(!)
⋮						

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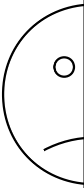

Outline

1. Root Data and Lie Algebras
    - Definition
    - A_1
    - G_2
2. Small Characteristic Trouble
    - Overview
    - Multidimensional Eigenspaces
3. Some Small Characteristic Solutions
4. Conclusions and Future Research

Small Characteristic Trouble

$$L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_{\alpha},$$

- $y, z \in Y$: $[y, z] = 0,$
- $y \in Y, \beta \in \Phi$: $[X_{\beta}, y] = \langle \beta, y \rangle X_{\beta},$
- $\alpha \in \Phi$: $[X_{-\alpha}, X_{\alpha}] = \alpha^{\vee},$
- $\alpha, \beta \in \Phi$: $[X_{\alpha}, X_{\beta}] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$

Trouble:

1. Multidimensional eigenspaces
2. $k + 1 \equiv 0$, so root chains are broken,
3. $k + 1 \equiv 0$, so $N_{\alpha,\beta} \equiv 0$,

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Multidimensional Eigenspaces

Steinberg, 1961

Complete list of multiplicities of roots, for root data of adjoint type

Cohen, R., 2008

Complete list of multiplicities of roots, for all root data

Char.	Root datum	Eigenspace dims
3	$A_2{}^{sc}$	3^2
3	G_2	$1^6, 3^2$
2	$A_3{}^{sc}, A_3^{(a)\star}$	4^3
2	$B_n{}^{ad}$ $(n \geq 2)$	$2^n, 4^{(n^2-n)/2}$
2	$B_2{}^{sc}$	4^2
2	$B_3{}^{sc}$	6^3
2	$B_4{}^{sc}$	$2^4, 8^3$
2	$B_n{}^{sc}$ $(n \geq 5)$	$2^n, 4^{(n^2-n)/2}$
2	$C_n{}^{ad}$ $(n \geq 3)$	$2n^1, 2^{n^2-n}$
2	$C_n{}^{sc}$ $(n \geq 3)$	$2n^1, 4^{(n^2-n)/2}$
2	$D_4^{(a),(b),(a+b)\star}$	4^6
2	$D_4{}^{sc}$	8^3
2	$D_n^{(a)\star}, D_n{}^{sc}$ $(n \geq 5)$	$4^{\binom{n}{2}}$
2	F_4	$2^{12}, 8^3$
2	G_2	4^3
2	all remaining cases	$2^N (N = \lvert\Phi^+\rvert)$

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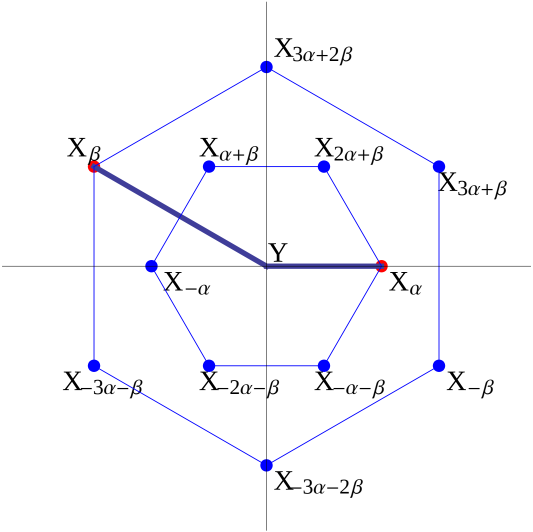
Outline

Example: Solving G_2 in char. 3



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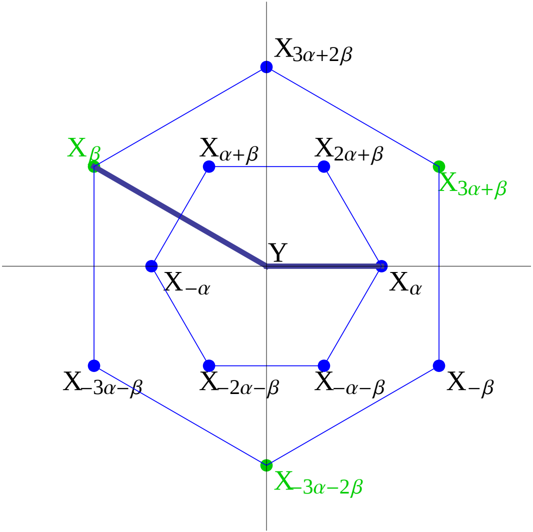

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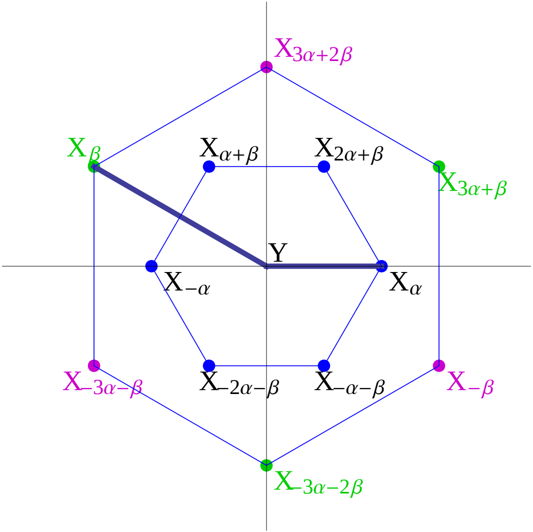
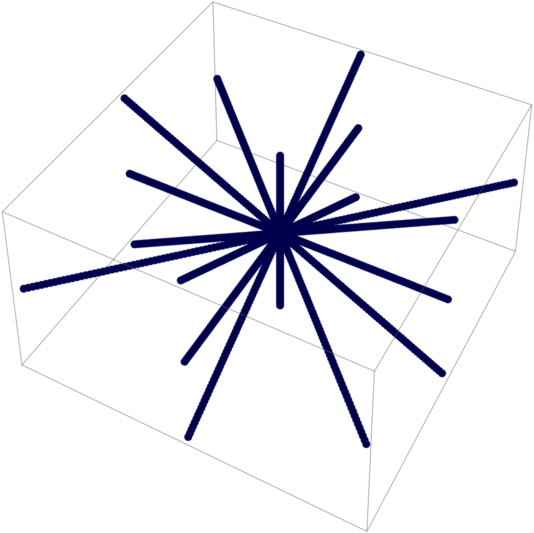


Example: Solving B_3 in char. 2



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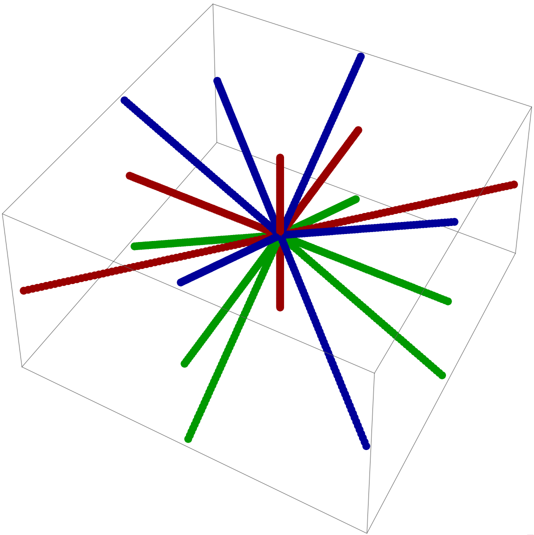

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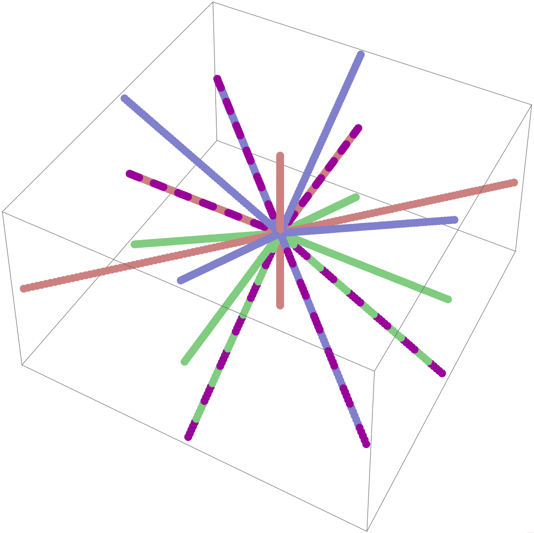

Some Small Characteristic Solutions

Solving many small puzzles,

- For $k + 1 \equiv 0$:
  - Find root chains in $[\cdot, \cdot]$ instead of eigentuples,
  - Small cases may be brute forced,
- For multidimensional eigenspaces:
  - "Pivot" using eigenspaces with smaller dimension,
  - Consider derivation algebra $\mathrm{Der}(L)$,
- General methods:
  - Many cases have nontrivial ideals, providing a reduction to a smaller case.

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Conclusion and Future Research

- Main challenges for computing Chevalley Bases:
  - Multidimensional eigenspaces
  - Broken root chains
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