Chevalley Bases over Fields of Small Characteristic

Dan Roozemond
Joint work with Arjeh Cohen

DIAMANT/EIDMA Symposium, 29/30 May 2008

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Eindhoven
University of Technology

Outline

1. Root Data and Lie Algebras
   - Definition
   - A_1
   - G_2
2. Small Characteristic Trouble
   - Overview
   - Multidimensional Eigenspaces
3. Some Small Characteristic Solutions
4. Conclusions and Future Research

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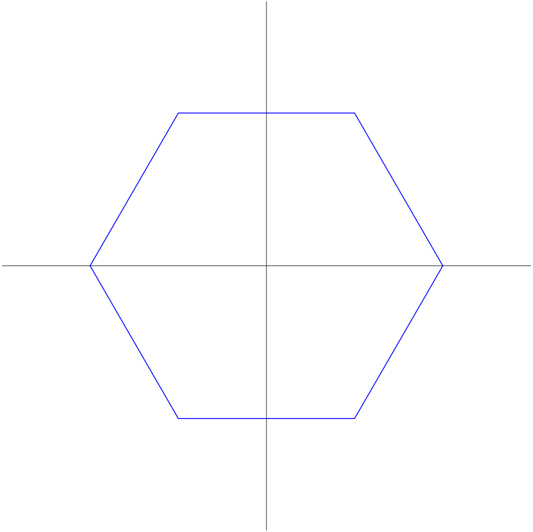
Root Systems



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- A root system of type A_2
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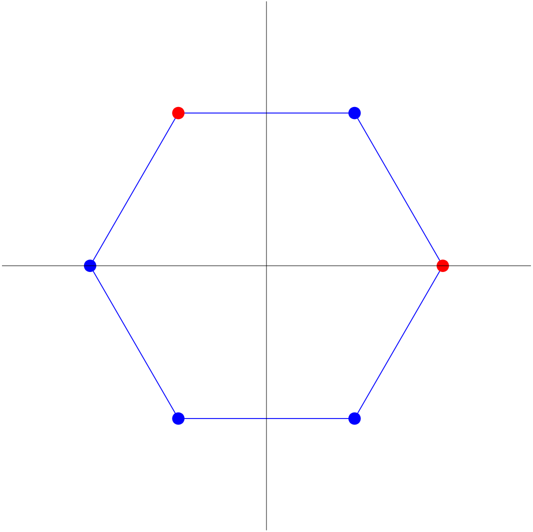

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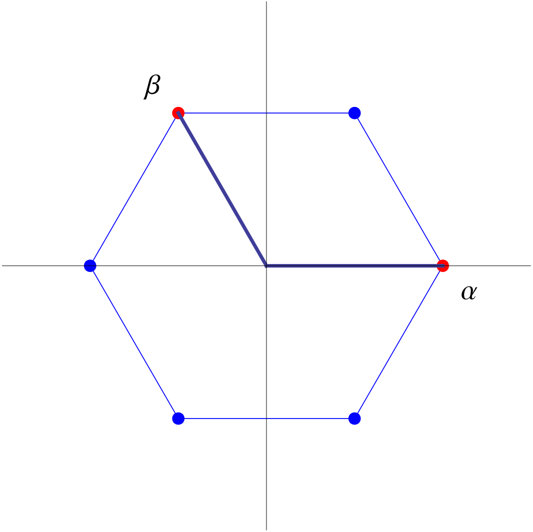

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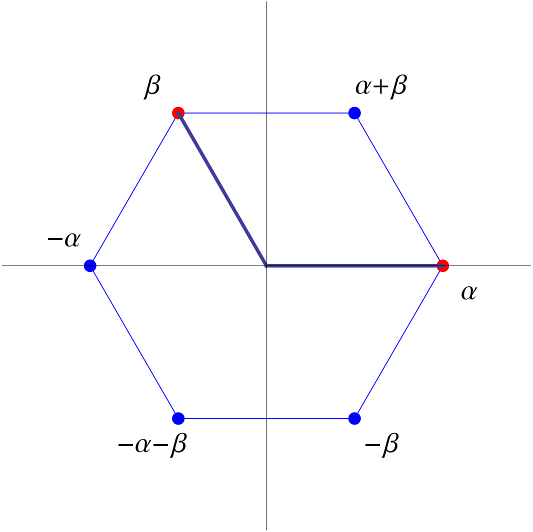

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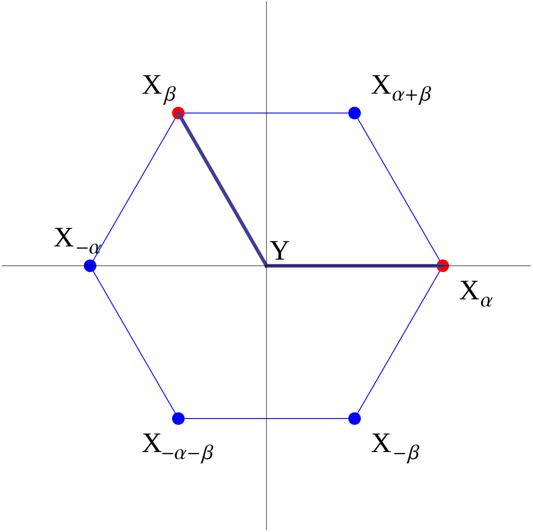

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Root Data

Definition (Root Datum)

$$R = (X, \Phi, Y, \Phi^\vee), \quad \langle\cdot,\cdot\rangle : X \times Y \to \mathbb{Z},$$

- X, Y: dual free $\mathbb{Z}$-modules,
- put in duality by $\langle\cdot,\cdot\rangle$,
- $\Phi \subseteq X$: roots,
- $\Phi^\vee \subseteq Y$: coroots.

A root system Φ induces several non-isomorphic root data, the "extremes" known as *adjoint* and *simply connected*.

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Lie Algebras

Definition (Chevalley Lie Algebra)

$$L_{\mathbb{Z}} = Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha,$$

with multiplication defined by

- $y, z \in Y$: $\quad [y, z] = 0,$
- $y \in Y, \beta \in \Phi$: $\quad [X_\beta, y] = \langle \beta, y \rangle X_\beta,$
- $\alpha \in \Phi$: $\quad [X_{-\alpha}, X_\alpha] = \alpha^\vee,$
- $\alpha, \beta \in \Phi$: $\quad [X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$

Such a basis is called a *Chevalley basis*.

Well known result: $N_{\alpha,\beta}$ can be chosen so that $N_{\alpha,\beta} = \pm(k+1)$, where k is the biggest number such that $\alpha - k\beta$ is a root.

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$L_{\mathbb{F}} = L_{\mathbb{Z}} \otimes \mathbb{F}$ gives a Lie algebra over $\mathbb{F}$.

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Finding Chevalley Bases:

Given: Lie algebra $L_{\mathbb{F}}$ with multiplication $[\cdot, \cdot]$,

Want: To find back (basis of) Y and X_α $(\alpha \in \Phi)$.

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Finding Chevalley Bases:

Given: Lie algebra $L_{\mathbb{F}}$ with multiplication $[\cdot, \cdot]$,
Root datum R, splitting Cartan subalgebra $Y \otimes \mathbb{F}$

Want: To find back (basis of) Y and X_{α} $(\alpha \in \Phi)$.

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Example: $\mathfrak{sl}_2$ / A_1

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[a, b] := ab - ba$$

	e	f	h
e	0	$-h$	$2e$
f	h	0	$-2f$
h	$-2e$	$2f$	0

$${A_1}^{\mathrm{sc}}: X = Y = \mathbb{Z},$$
$$\Phi = \{\alpha = 2, -\alpha = -2\},$$
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$$\begin{aligned} L_{\mathbb{Z}} &= Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha \\ &= \mathbb{Z} y \oplus \mathbb{Z} X_\alpha \oplus \mathbb{Z} X_{-\alpha}, \end{aligned}$$

$$\begin{aligned} [X_\alpha, X_{-\alpha}] &= -\alpha^\vee = -y, \\ [X_\alpha, y] &= \langle \alpha, y \rangle X_\alpha = 2X_\alpha, \\ \dots &= \dots \end{aligned}$$

	X_α	$X_{-\alpha}$	y
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Example 2: G_2

	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$	
X_α	$2X_\alpha$	$-X_\alpha$	2	-1
X_β	$-3X_\beta$	$2X_\beta$	-3	2
$X_{\alpha+\beta}$	$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	-1	1
$X_{2\alpha+\beta}$	$X_{2\alpha+\beta}$	0	1	0
$X_{3\alpha+\beta}$	$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	3	-1
$X_{3\alpha+2\beta}$	0	$X_{3\alpha+2\beta}$	0	1

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$X_{2\alpha+\beta}$	$X_{2\alpha+\beta}$	0	1	0
$X_{3\alpha+\beta}$	$3X_{3\alpha+\beta}$	$-X_{3\alpha+\beta}$	3	-1
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$X_{-\alpha}$	$-2X_{-\alpha}$	$X_{-\alpha}$	−2	1
$X_{-\beta}$	$3X_{-\beta}$	$-2X_{-\beta}$	3	−2
$X_{-\alpha-\beta}$	$X_{-\alpha-\beta}$	$-X_{-\alpha-\beta}$	1	−1
$X_{-2\alpha-\beta}$	$-X_{-2\alpha-\beta}$	0	−1	0
$X_{-3\alpha-\beta}$	$-3X_{-3\alpha-\beta}$	$X_{-3\alpha-\beta}$	−3	1
$X_{-3\alpha-2\beta}$	0	$-X_{-3\alpha-2\beta}$	0	−1

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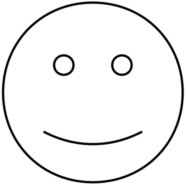

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Example 2: G_2

	$\alpha^\vee$	$\beta^\vee$	$\mathbb{Z}$		$GF(3^m)$		
X_α	$2X_\alpha$	$-X_\alpha$	2	-1	-1	-1	
X_β	$-3X_\beta$	$2X_\beta$	-3	2	0	-1	(!)
$X_{\alpha+\beta}$	$-X_{\alpha+\beta}$	$X_{\alpha+\beta}$	-1	1	-1	1	
$X_{2\alpha+\beta}$	$X_{2\alpha+\beta}$	0	1	0	1	0	
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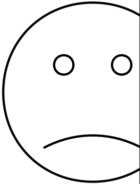

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Trouble:

1. Multidimensional eigenspaces
2. $k + 1 \equiv 0$, so root chains are broken,
3. $k + 1 \equiv 0$, so $N_{\alpha,\beta} \equiv 0$,

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Multidimensional Eigenspaces

Steinberg, 1961

Complete list of multiplicities of roots, for root data of adjoint type

Cohen, R., 2008

Complete list of multiplicities of roots, for all root data

Char.	Root datum	Eigenspace dims
3	$A_2{}^{sc}$	3^2
3	G_2	$1^6, 3^2$
2	$A_3{}^{sc}, A_3^{(a)\star}$	4^3
2	$B_n{}^{ad}$ $(n \geq 2)$	$2^n, 4^{(n^2-n)/2}$
2	$B_2{}^{sc}$	4^2
2	$B_3{}^{sc}$	6^3
2	$B_4{}^{sc}$	$2^4, 8^3$
2	$B_n{}^{sc}$ $(n \geq 5)$	$2^n, 4^{(n^2-n)/2}$
2	$C_n{}^{ad}$ $(n \geq 3)$	$2n^1, 2^{n^2-n}$
2	$C_n{}^{sc}$ $(n \geq 3)$	$2n^1, 4^{(n^2-n)/2}$
2	$D_4^{(a),(b),(a+b)\star}$	4^6
2	$D_4{}^{sc}$	8^3
2	$D_n^{(a)\star}, D_n{}^{sc}$ $(n \geq 5)$	$4^{\binom{n}{2}}$
2	F_4	$2^{12}, 8^3$
2	G_2	4^3
2	all remaining cases	$2^N (N = \|\Phi^+\|)$

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Outline

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Some Small Characteristic Solutions

Solving many small puzzles,

- For $k + 1 \equiv 0$:
  - Find root chains in $[\cdot, \cdot]$ instead of eigentuples,
  - Small cases may be brute forced,
- For multidimensional eigenspaces:
  - "Pivot" using eigenspaces with smaller dimension,
  - Consider derivation algebra $\mathrm{Der}(L)$,
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  - Many cases have nontrivial ideals, providing a reduction to a smaller case.

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Conclusion and Future Research

- Main challenges for computing Chevalley Bases:
  - Multidimensional eigenspaces
  - Broken root chains
- Found solutions for majority of the cases,
- Bigger picture:
  - Automorphisms of Lie algebras,
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Example: $A_1{}^{sc}$ / $A_1{}^{ad}$

$A_1{}^{sc}$: $X = Y = \mathbb{Z}$,
$\Phi = \{\alpha = 2, -\alpha = -2\}$,
$\Phi^\vee = \{\alpha^\vee = 1, -\alpha^\vee = -1\}$,

$$\begin{aligned} L_\mathbb{Z} &= Y \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z}X_\alpha \\ &= \mathbb{Z}y \oplus \mathbb{Z}X_\alpha \oplus \mathbb{Z}X_{-\alpha}, \end{aligned}$$

$$\begin{aligned} [X_\alpha, X_{-\alpha}] &= -\alpha^\vee = -y, \\ [X_\alpha, y] &= \langle \alpha, y \rangle X_\alpha = 2X_\alpha, \\ \dots &= \dots \end{aligned}$$

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Observe:

- $y \mapsto \frac{1}{2}y$ maps $A_1{}^{sc}$ to $A_1{}^{ad}$,
- Except if the characteristic is 2!

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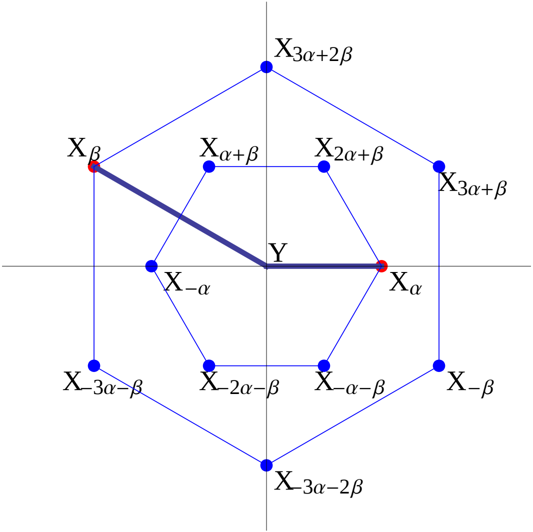
Example: Solving G_2 in char. 3



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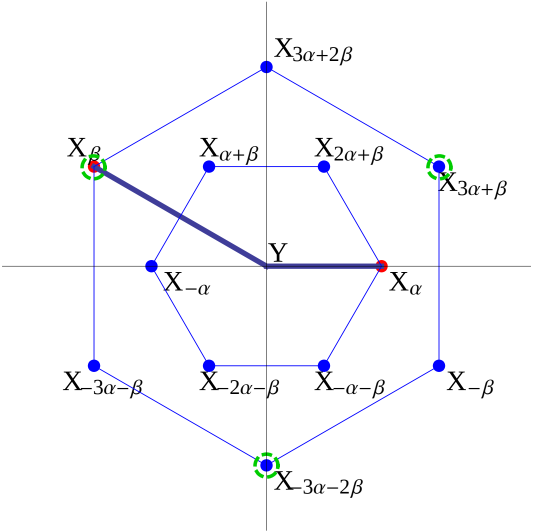

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