Conjugation in Groups of Lie type

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Contents

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G a reductive algebraic group over the field K.

- Fix a maximal torus T of G,
- and let $(X, \Phi, Y, \Phi^\star)$ be its root datum,
- and W its Weyl group.

Steinberg presentation

Generators

- $x_\alpha(a)$, for $\alpha \in \Phi$ and $a \in K$,
- $y \otimes t$, for $y \in Y$ and $t \in K^*$.

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- $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$.

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Subgroups:

- T, generated by $y \otimes t$,
- N, generated by n_α,
- U, generated by $x_\alpha(a)$, with $\alpha \in \Phi^+$,
- U^-, generated by $x_\alpha(a)$, with $\alpha \in \Phi^-$,
- $B = TU$.

Representations

Fix a *rational representation*

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Weights :

$$V_\mu = \{v \in V \mid v(\gamma \otimes t) = t^{\langle \mu, \gamma \rangle} v \text{ for all } \gamma \in Y, t \in K^*\},$$

now $V = \bigoplus V_\mu$.

Weight Base

Definitions:

- n is the rank of the root system of G,
- $\delta : G \to G$: $x_\alpha(a) \mapsto x_{-\alpha}(-a)^{-1}$,
- $I \subseteq \{1, \ldots, n\}$: $W_I = \langle n_{\alpha_i} \mid i \in I \rangle$,
- "Levi complement": $L_I = BW_IB \cap B^\delta W_I^\delta B^\delta$.

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Calculating a weight base

(Cohen, Murray, Taylor, 2003)

In: Reductive algebraic group G over K,
Highest weight representation $\rho : G \to \mathrm{GL}(V)$,

Out: $\lambda_1, \ldots, \lambda_k \in V$, $J_1, \ldots, J_k \subseteq \{1, \ldots, n\}$,
such that $G = L_{J_1} \geq L_{J_2} \geq \ldots \geq L_{J_{k+1}} = T$,
and λ_i is a highest weight for L_{J_i} acting on V.

Original

Row and column reduction

(Cohen, Murray, Taylor, 2003)

In: Reductive algebraic group G over K,
Representation $\rho : G \to \mathrm{GL}(V)$,
$A \in \rho(G)$.

Out: $g \in G$ such that $A = \rho(g)$.

Algorithm: A generalization of row and column reduction.

For vectors (I)

Row reduction for vectors (I)

In: Reductive algebraic group G over K,
Representation $\rho : G \longrightarrow \mathrm{GL}(V)$,
$\{z_1, \ldots, z_q\} \subseteq V$,

Out: $g \in G$ such that, for all $i \in \{1, \ldots, q\}$:

$$z_i g^{-1} \in \bigoplus_{\mu \succeq \lambda_i} V_\mu.$$

Into Borel

Conjugation into the Borel subgroup

In: Reductive algebraic group G over K,
c a semisimple element of G,

Out: $d \in B, a \in G$ such that $d = c^a$

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Sketch of the algorithm:

1. Compute centraliser $\mathfrak{c}$ of c in the Lie algebra $\mathfrak{g}$,
2. Compute splitting Cartan subalgebra $\mathfrak{h}$ contained in $\mathfrak{c}$,
3. Write down basis of $\mathfrak{g}$ with respect to $\mathfrak{h}$,
4. Row reduce some entries of this basis, obtaining $g \in G$,
5. g^{-1} does the job.

Into Borel: Example

We consider A_3 over $\mathbb{F}_{59}$.

- $$c = \begin{pmatrix} 9 & 1 & 53 & 8 \\ 42 & 0 & 8 & 44 \\ 48 & 0 & 7 & 23 \\ 12 & 1 & 23 & 8 \end{pmatrix}$$

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- Centraliser is 5 dimensional over $\mathbb{F}_{59}$,
- Cartan Subalgebra is 3 dimensional over $\mathbb{F}_{59}^4$,
- We row reduce:

$$\begin{pmatrix} 4 & 25 & 32 & 18 & 23 & 5 & 35 & 39 & 9 & 54 & 52 & 44 & 20 & 29 & 52 \\ 55 & 34 & 0 & 41 & 0 & 2 & 49 & 17 & 24 & 30 & 9 & 0 & 57 & 47 & 42 \end{pmatrix}$$

Into Borel: Example

We consider A_3 over $\mathbb{F}_{59}$.

- And find

$$g^{-1} = \begin{pmatrix} 0 & 0 & 0 & 58 \\ 0 & 0 & 1 & 21 \\ 0 & 58 & 50 & 25 \\ 1 & 51 & 30 & 53 \end{pmatrix}$$

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$$g^{-1} = \begin{pmatrix} 0 & 0 & 0 & 58 \\ 0 & 0 & 1 & 21 \\ 0 & 58 & 50 & 25 \\ 1 & 51 & 30 & 53 \end{pmatrix}$$

- Now

$$c' = gcg^{-1} = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 7 & 28 & 0 & 0 \\ 24 & 44 & 50 & 0 \\ 8 & 1 & 0 & 50 \end{pmatrix}$$

Towards the standard torus

- $$c' = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 7 & 28 & 0 & 0 \\ 24 & 44 & 50 & 0 \\ 8 & 1 & 0 & 50 \end{pmatrix},$$
- Recall $\delta : G \to G$: $x_\alpha(a) \mapsto x_{-\alpha}(-a)^{-1}$,

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- Recall $\delta : G \to G$: $x_\alpha(a) \mapsto x_{-\alpha}(-a)^{-1}$,
- Now
$$c'^{\delta} = \begin{pmatrix} 26 & 36 & 31 & 53 \\ 0 & 19 & 12 & 48 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$
- So we would like to do the same algorithm again...

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- So we would like to do the same algorithm again...
- but that will not work.

For vectors (II)

Row reduction for vectors (II)

In: Reductive algebraic group G over K,
Representation $\rho : G \to \mathrm{GL}(V)$,
$\{z_1, \ldots, z_q\} \subseteq V$,

Out: $g \in G$ such that, for all $i \in \{1, \ldots, q\}$:

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In: Reductive algebraic group G over K,
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Out: $g \in G$ such that, for all $i \in \{1, \ldots, q\}$:

$$z_i g^{-1} \in \bigoplus_{\mu \succeq \lambda_i} V_\mu$$

and, for $i \in \{1, \ldots, k\}, j \in \{1, \ldots, i-1\}$,

$$z_i \cdot v_i \neq 0 \text{ and } z_i \cdot v_j = 0 \Rightarrow g \in B^\delta.$$

/ department of mathematics and computer science

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Sketch of the algorithm:

1. Compute $c' \in B$,
2. Conjugate c'^{δ} into B as before, but use new row reduction,
3. Compose results.

Into standard torus: Example

- Now

$$c'^{\delta} = \begin{pmatrix} 26 & 36 & 31 & 53 \\ 0 & 19 & 12 & 48 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

Into standard torus: Example

- Now

$$c'^{\delta} = \begin{pmatrix} 26 & 36 & 31 & 53 \\ 0 & 19 & 12 & 48 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

- We row reduce:

$$\begin{pmatrix} 1 & 14 & 0 & 0 & 0 & 45 & 34 & 19 & 13 & 25 & 0 & 0 & 0 & 54 & 19 \\ 9 & 8 & 42 & 0 & 57 & 38 & 1 & 58 & 58 & 0 & 0 & 15 & 0 & 22 & 0 \end{pmatrix}$$

Into standard torus: Example

- Now

$$c'^{\delta} = \begin{pmatrix} 26 & 36 & 31 & 53 \\ 0 & 19 & 12 & 48 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

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- And obtain g', such that

$$g'c'^{\delta}g'^{-1} = \begin{pmatrix} 33 & 0 & 0 & 0 \\ 0 & 40 & 0 & 0 \\ 0 & 0 & 46 & 0 \\ 0 & 0 & 0 & 46 \end{pmatrix}$$

Weighted Dynkin Diagrams

“Definition”

In: Lie algebra $\mathfrak{g}$,
with simple root system Δ,
$e \in \mathfrak{g}$ nonzero nilpotent,

Out: $p_\alpha \in \{0, 1, 2\}$, $\alpha \in \Delta$.

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with simple root system Δ,
$e \in \mathfrak{g}$ nonzero nilpotent,

Out: $p_\alpha \in \{0, 1, 2\}$, $\alpha \in \Delta$.

Why weighted Dynkin diagrams are useful

Theorem: e and e' have the same weighted Dynkin diagram if and only if they are in the same G-orbit.

Computing Weighted Dynkin Diagrams

Let $e \in \mathfrak{g}$, and $\langle e, h, f \rangle$ the $\mathfrak{sl}_2$ subalgebra containing e.

Computing

Computing the weighted Dynkin diagram for e is easy if h in the standard torus $\mathfrak{t}$.

Computing Weighted Dynkin Diagrams

Example in Magma (I)

```
> G := GroupOfLieType("D4", Rationals());
> rho,L := AdjointRepresentation(G);
> rhoL := AdjointRepresentation(L);
>
> e1 := L.1;
> lbl1 := WeightedDynkinDiagramLabels(e1);
> lbl1;
[ 0, 1, 0, 0 ]
```

Computing Weighted Dynkin Diagrams

Example in MAGMA (II)

```
> lbl1;
[ 0, 1, 0, 0 ]
> c := elt<G | [* <1,3>, <2,3>, 2, 3 *] >; c;
x2(3) x5(9) x1(3) n2 n3
> C := ChangeRing(rho(c), BaseRing(L));
>
> e2 := L!(e1*C);
> e2;
(-3 0 0 1 0 ....... 0 0 0)
>
> lbl2 := WeightedDynkinDiagramLabels(e2);
> lbl2;
[ 0, 1, 0, 0 ]
```

Conclusion

- New variants of row reduction,
- Application: Conjugation,
- Application: Weighted Dynkin diagrams.

Future Research

- Small characteristic cases,
- Fix bugs,
- Find conjugators for subalgebras in the same orbit,
- ...

Questions?