Combinatorics with Representations
or
Embedding LiE in Magma

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- Embedding LiE in Magma

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Lie Groups: Introduction - I

A Lie group is a group that also has the structure of a differentiable manifold, such that the maps of multiplication and inversion are differentiable maps.

Lie Groups: Introduction - II

Restrict to *connected reductive complex* Lie groups:

- Nice Classification Theory
- Nice Representation Theory

Lie Groups: Classification Theory - I

Bourbaki, 1975: if g is a connected reductive complex Lie group:

$$g' = \otimes \begin{cases} \text{simply connected semisimple complex grp} \\ \text{complex torus} \end{cases}$$

and

$$g = \xi(g'), \xi \text{ a homomorphism with finite kernel.}$$

Lie Groups: Classification Theory - I

Bourbaki, 1975: if g is a connected reductive complex Lie group:

$$g' = \otimes \left\{ \begin{array}{l} \text{simply connected semisimple complex grp} \;\otimes \left\{ \begin{array}{c} \text{s.c. simple grp} \\ \vdots \\ \text{s.c. simple grp} \end{array} \right. \\ \\ \text{complex torus} \end{array} \right.$$

and

$$g = \xi(g'), \xi \text{ a homomorphism with finite kernel.}$$

Lie Groups: Classification Theory - II

Simply connected simple Lie groups:

- Classical types: A_n, B_n, C_n, D_n,
- Exceptional types: F_4, G_2, E_6, E_7, E_8.

Lie Groups: Classification Theory - III

We restrict to

$$g = \otimes \left\{ \begin{array}{l} \text{simply connected semisimple complex grp} \quad \otimes \left\{ \begin{array}{c} \text{s.c. simple grp} \\ \vdots \\ \text{s.c. simple grp} \end{array} \right. \\ \\ \text{complex torus} \end{array} \right. .$$

Then, e.g.

$$g = A_4 C_3 B_1 T_2.$$

Lie Groups: Weights - I

Fix a maximal torus in g

$$T = S \otimes T'$$

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Weights: elements of $\Lambda(T)$ = algebraic group morphisms $T \to \mathbb{C}^\star$.
Weights: isomorphism classes of 1-dimensional T-modules.

Lie Groups: Weights - I

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$$\mathrm{rank}(g) := \dim(T)$$

Weights: elements of $\Lambda(T)$ = algebraic group morphisms $T \longrightarrow \mathbb{C}^\star$.
Weights: isomorphism classes of 1-dimensional T-modules.
Weights: elements of $\mathbb{C}^r$.

Representations - I

- $$\rho : g \to \mathrm{GL}(V), \qquad \rho \text{ a Lie group homomorphism.}$$

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- Equivalently, we specify a left action of g on V, such that each map

  $$v \mapsto x.v, \qquad v \in V, x \in g,$$

  is linear and depends on a differentiable way on x. Then V is called a *g-module*.
- V is called *irreducible* if: $g.V' = V' \Rightarrow V' = 0$ or $V' = V$.

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Lie Groups: Weights - II / Representations - II

The torus T is reductive and Abelian, so it is diagonalisable in any g-representation,

so

If M is a g-**module**, then the restriction of M to T is a direct sum of 1-dimensional T-modules, and therefore described by a set of **weights** with **multiplicities**.

Lie Groups: Weights - III / Representations - III

- *Adjoint representation* of g: its representation on the Lie algebra of g,
- *Root system* Φ: Non-zero weights occurring in the adjoint representation,
- *Fundamental roots*: $\{\alpha_1, \ldots, \alpha_s\} \subset \Phi$.

Lie Groups: Weights - III / Representations - III

- *Adjoint representation* of g: its representation on the Lie algebra of g,
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Partial ordering of weights:

$$v \prec v' \Leftrightarrow v - v' = \sum_{i=1}^{s} \kappa_i \alpha_i, \qquad \kappa_i \in \mathbb{N}_0.$$

Representations - III

Two facts:

- Every g-module decomposes as a direct sum of irreducible modules,
- Irreducible representations $\Leftrightarrow \Lambda^+(T)$: *dominant weights*:

Assign to each irreducible module its highest weight.

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- Representations $\Leftrightarrow$ *Decomposition Polynomials*.

Representations - IV: Summary

Irreducible Modules

$\Updownarrow$

$\Lambda(T)$: Weights
and Multiplicities

$\Updownarrow$

$\Lambda^+(T)$:
Dominant Weights
and Multiplicities

$\Updownarrow$

Unique
Highest Weight

Representations - IV: Summary

	Module
	$\Updownarrow$
Irreducible Modules	$\oplus$ Irreducible
$\Updownarrow$	$\Updownarrow$
$\Lambda(T)$: Weights and Multiplicities	$\cdots$
$\Updownarrow$	$\Updownarrow$
$\Lambda^+(T)$: Dominant Weights and Multiplicities	$\cdots$
$\Updownarrow$	$\Updownarrow$
Unique Highest Weight	Highest Weights and Multiplicities

Representations - IV: Summary

	Module	**LiE**
	$\Updownarrow$	
Irreducible Modules	$\bigoplus$ Irreducible	
$\Updownarrow$	$\Updownarrow$	
$\Lambda(T)$: Weights and Multiplicities	$\cdots$	*Character Polynomial*
$\Updownarrow$	$\Updownarrow$	
$\Lambda^+(T)$: Dominant Weights and Multiplicities	$\cdots$	*Dominant Character Polynomial*
$\Updownarrow$	$\Updownarrow$	
Unique Highest Weight	Highest Weights and Multiplicities	*Decomposition Polynomial*

Example: Tensor - I

```
> A2 := RootDatum("A2" : Isogeny := "SC");
> L := LieAlgebra(A2, Rationals());
> _,_,T := StandardBasis(L);
> rho := AdjointRepresentation(L);
> V := VectorSpace(Rationals(), 8);
/* Now V is an L-module by v --> x.v = rho(x).v */
```

Example: Tensor - II

```
> rho := AdjointRepresentation(L);
> V := VectorSpace(Rationals(), 8);
/* Now V is an L-module by v --> x.v = rho(x).v */
> MT := [ Transpose(Matrix(rho(t))) : t in T ]; MT;
```

$$\begin{pmatrix} -1 &&&&&&& \\ & 1 &&&&&& \\ && -2 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 2 && \\ &&&&&& -1 & \\ &&&&&&& 1 \end{pmatrix}, \begin{pmatrix} -1 &&&&&&& \\ & -2 &&&&&& \\ && 1 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& -1 && \\ &&&&&& 2 & \\ &&&&&&& 1 \end{pmatrix}$$

Example: Tensor - III

```
> rho := AdjointRepresentation(L);
> V := VectorSpace(Rationals(), 8);
/* Now V is an L-module by v --> x.v = rho(x).v */
> MT := [ Transpose(Matrix(rho(t))) : t in T ]; MT;
```

$$\begin{pmatrix} -1 &&&&&&& \\ & 1 &&&&&& \\ && -2 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 2 && \\ &&&&&& -1 & \\ &&&&&&& 1 \end{pmatrix}, \begin{pmatrix} -1 &&&&&&& \\ & -2 &&&&&& \\ && 1 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& -1 && \\ &&&&&& 2 & \\ &&&&&&& 1 \end{pmatrix}$$

$\Rightarrow$ Weights are:

$$(-1,-1),(1,-2),(-2,1),(0,0),(0,0),(2,-1),(-1,2),(1,1)$$

Example: Tensor - IV

$\Rightarrow$ Weights are:

$$(-1,-1),(1,-2),(-2,1),(0,0),(0,0),(2,-1),(-1,2),(1,1)$$

$\Rightarrow$ Dominant weights are:

$$(0,0)\text{ (twice) },(1,1).$$

Example: Tensor - V

Lie group $g = A_2$:

- ρ_1: Adjoint representation of g, dimension 8,
- ρ_2: Irreducible Representation with highest weight $(2, 0)$, dimension 6.

Tensoring these representations will give a representation of dimension 48.

Example: Tensor - VI

Tensoring these representations will give a representation of dimension 48.

```
> rho1 := HighestWeightRepresentation(L, [1,1]);
> rho2 := HighestWeightRepresentation(L, [2,0]);
> MT := [ Transpose(TensorProduct(Matrix(rho1(t)),
        Matrix(rho2(t)))) : t in T ] ;
```

Weights:

```
> {* Vector([(B[i]*m)[i] : m in MT]):i in [1..#B] *};
{*
     ( 0 -1)^^2,
     (-2  0)^^3,
       ...
     ( 0 -2)^^3,
     (4 2)
*}
```

Example: Tensor - VII

```
> drho1 := LieRepresentationDecomposition(A2, [1,1]);
> drho2 := LieRepresentationDecomposition(A2, [2,0]);
> dtp := Tensor(drho1, drho2);
> dtp:Maximal;
Highest weight decomposition of representation of:
     A2: Simply connected root datum of dimension 2 of ty
     Weights: [
               (3 1),
               (0 1),
               (2 0),
               (1 2)
          ]
     Multiplicities: [ 1, 1, 1, 1 ]
> Dim(dtp);
48
```

Example: Tensor - VIII

```
> dtp:Maximal;
Highest weight decomposition of representation of:
     A2: Simply connected root datum of dimension 2 of ty
     Weights: [
               (3 1),
               (0 1),
               (2 0),
               (1 2)
          ]
     Multiplicities: [ 1, 1, 1, 1 ]
> Dim(dtp);
48
> [ Dim(LieRepresentationDecomposition(A2, w)) :
                  w in [[3,1],[0,1],[2,0],[1,2]] ];
[ 24, 3, 6, 15 ]
```

LiE - I

- van Leeuwen, Cohen, Lisser, 1992
- Advantages:
  - Specifically for Lie group computations,
  - $\Rightarrow$ Extremely fast.
- Disadvantages:
  - Specifically for Lie group computations,
  - More or less unchanged since 2000,
  - Not very customizable.

LiE - II

1. Lie groups
2. Root systems
3. The Weyl group
4. Operations related to the Symmetric group
5. Representations

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1. Lie groups
2. Root systems
3. The Weyl group
4. Operations related to the Symmetric group
5. Representations
   - Tensor
   - Adams operator
   - Alternating Weyl sum
   - Symmetric / Alternating Tensor
   - Branch / Collect
   - ...

Embedding LiE in Magma - I

- No black-box approach,

Embedding LiE in Magma - I

- No black-box approach,
- Connected reductive complex Lie groups ⇒ Root data,
- Existing functionality by Murray, Taylor, de Graaf, Haller,
- Focus on representations,
- First a package (intrinsics),
- Port critical parts to C.

Embedding LiE in Magma - II: Timings

Tensoring two (non-irreducible) representations:

Group	#	Dims	LiE	Magma (package)	Magma (C)
A3	10	360, 444	0.006		
A3	10	100, 40	0.007		
D5	10	175, 664	0.005		
E8	3	60760, 11625	0.020		
E8	1	8192000, 34537472000	0.100		

Embedding LiE in Magma - II: Timings

Tensoring two (non-irreducible) representations:

Group	#	Dims	LiE	Magma (package)	Magma (C)
A3	10	360, 444	0.006	0.302	
A3	10	100, 40	0.007	0.612	
D5	10	175, 664	0.005	0.299	
E8	3	60760, 11625	0.020	6.823	
E8	1	8192000, 34537472000	0.100	n/a	

Embedding LiE in Magma - II: Timings

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Group	#	Dims	LiE	Magma (package)	Magma (C)
A3	10	360, 444	0.006	0.302	0.095
A3	10	100, 40	0.007	0.612	0.055
D5	10	175, 664	0.005	0.299	0.104
E8	3	60760, 11625	0.020	6.823	0.370
E8	1	8192000, 34537472000	0.100	n/a	4.660

Conclusion & Todo

Done:

- All functionality of LiE in a Magma package,
- Created Magma type for decomposition polynomials,
- Ported some speed-critical functions in C.

Conclusion & Todo

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- All functionality of LiE in a Magma package,
- Created Magma type for decomposition polynomials,
- Ported some speed-critical functions in C.

Todo:

- Port a few more speed-critical functions to C,
- Speed up existing DominantCharacter functionality,
- Write documentation,
- Work on *maximal subgroups of algebraic groups*...

Questions?