Lie Algebras generated by Extremal Elements

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What is the successor of the series

$$1, 3, 8, 28, \ldots$$

$$1, 3, 8, 28, 537$$

What is the successor of the series

$$0, 3, 8, 28, \ldots$$

$$0, 3, 8, 28, 248$$

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- Motivation
- Introduction to Lie Algebras
- Extremal Elements + Examples:

  $\longrightarrow$ One generator case

  $\longrightarrow$ Two generator case
- Some existing results
- Semi-simplicity theorem

  $\longrightarrow$ Five generator case
- Examples
- Three theorems and three conjectures
- Conclusion

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- Forget about derivations and define 'Lie Algebra'

Lie Algebras - 1

A Lie algebra L is a vector space with a map $[\cdot,\cdot] : V \times V \to V$ such that

1. $[\cdot,\cdot]$ is bilinear,
2. $[\cdot,\cdot]$ is skew-symmetric:

$$[x, x] = 0 \text{ for all } x \in L, \text{ and}$$

3. $[\cdot,\cdot]$ satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in L.$$

Note that in characteristic not 2, property 2. is equivalent to $[x, y] = -[y, x]$.

Lie Algebras - 2: An example

An example: $\mathfrak{gl}(V)$.

Let V be a vector space, and L the ring of linear transformations $V \to V$. Define $[x, y] = xy - yx$ for $x, y \in L$.

Lie Algebras - 3: Another example

The Lie algebra A_1 consists of three elements, such that:

- $[x, y] = z$,
- $[x, z] = [x, [x, y]] = -2x$, and
- $[y, z] = [y, [x, y]] = 2y$.

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	x	**y**	**z**
x	0	z	$-2x$
y	$-z$	0	$2y$
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Representation: Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $[x, y] = xy - yx$.

Then $z = [x, y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Lie Algebras - 4: Ideals

- Ideal: A subspace I is an ideal of L if

$$x \in L, y \in I \Rightarrow [x, y] \in I.$$

- Simplicity: L is called simple if it has no ideals except $\{0\}$ and L.
- Semi-simplicity

Lie Algebras - 5: Ideals

- Solvability: I is solvable if the sequence

$$I, [I, I], [[I, I], [I, I]], [[[I, I], [I, I]], [[I, I], [I, I]]], \ldots$$

  'goes to zero'.
- Radical: $\mathrm{Rad}(L)$ is the largest solvable ideal.
- Theorem: $L/\mathrm{Rad}(L)$ is semi-simple.

Lie Algebras - 6: Simple Lie algebras

The classical Lie algebras:

- A_n ($n \geq 1$): $\mathfrak{sl}_{n+1}$, of dimension $(n+1)^2 - 1$,
- B_n ($n \geq 2$): $\mathfrak{o}_{2n+1}$, of dimension $n(2n+1)$,
- C_n ($n \geq 3$): $\mathfrak{sp}_{2n}$, of dimension $n(2n+1)$,
- D_n ($n \geq 4$): $\mathfrak{o}_{2n}$, of dimension $n(2n-1)$.

The exceptional Lie algebras: G_2, F_4, E_6, E_7, E_8.

Extremal Elements: Definition

$x \in L$ is called *extremal* if for all $y \in L$, we have

$$[x, [x, y]] = \alpha x \text{ for some } \alpha \in \mathbb{F}.$$

Lie Algebras Generated by One Extremal Element

- Lie algebra L over the field $\mathbb{F}$ generated by extremal element x,
- $[x, x] = 0$, so $L = \mathbb{F}x$,
- $\dim(L) = 1$,
- L is solvable.

Lie Algebras Generated by Two Extremal Elements

- L, generated by extremal elements x and y,
- $[x,[x,y]] = \alpha x$ and $[y,[y,x]] = \beta y$.

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x	0	z	αx
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- If $[x,y] = 0$: $\dim(L) = 2$ and L is solvable,
- If $[x,y] \neq 0$ and $\alpha = 0$: $\dim(L) = 3$ and L is solvable ('Heisenberg'),
- If $[x,y] \neq 0$ and $\alpha \neq 0$: $L \cong A_1$!

Some results of CSUW01 - 1

Cohen, Steinbach, Ushirobira, Wales, 2001. Suppose L is generated by finitely many extremal elements. Then:

- L is finite dimensional, and
- L is spanned by extremal elements.

(Using [ZK90])

Some results of CSUW01 - 2

# Extr Elts	Dimension	General Lie algebra
1	1	n/a (0 dimensional)
2	3	A_1 (3 dimensional)
3	8	A_2 (8 dimensional)
4	28	?
5	537	?

$$1, 3, 8, 28, 537$$

Some results of CSUW01 - 3

Simple Lie algebra	# Extr Elts	Condition
A_n	$n+1$	$n \geq 1$
B_n	$n+1$	$n \geq 3$
C_n	$2n$	$n \geq 2$
D_n	n	$n \geq 4$
G_2	4	
E_6, E_7, E_8, F_4	5	

Verification using GAP

# Extr Elts	Dimension	General Lie algebra
1	1	n/a (0 dimensional)
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4	28	D_4 (28 dimensional)
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Semi-Simplicity Theorem - 1

If

- L_1 is a semi-simple Lie algebra generated by n extremal elements and no fewer, and
- L_2 is a semi-simple Lie algebra generated by m extremal elements and no fewer,

then

- $L_1 + L_2$ is generated by $n + m$ extremal elements,
- and no fewer.

Semi-Simplicity Theorem - 2: Examples

- The simple Lie algebra A_1 requires 2 extremal elements,
- the simple Lie algebra A_2 requires 3 extremal elements,
- the simple Lie algebra G_2 requires 4 extremal elements, and

so

- the semi-simple Lie algebra $A_1 + A_1$ requires 4 extremal elements, and
- the semi-simple Lie algebra $A_2 + G_2$ requires 7 extremal elements.

Five Generators - 1

What if the 537-dimensional Lie algebra generated by five extremal elements were semi-simple?

Simple Lie algebra	Dimension	# Extr Elts	Condition
A_n	$(n+1)^2-1$	$n+1$	$n \geq 1$
B_n	$n(2n+1)$	$n+1$	$n \geq 3$
C_n	$n(2n+1)$	$2n$	$n \geq 2$
D_n	$n(2n-1)$	n	$n \geq 4$
G_2	14	4	
E_6	78	5	
E_7	133	5	
E_8	248	5	
F_4	52	5	

Five Generators - 2

What if the 537-dimensional Lie algebra generated by five extremal elements were semi-simple?

Lie Algebra	Dim	# Extr Elts
$A_1 = B_1 = C_1$	3	2
A_2	8	3
$B_2 = C_2$	10	4
G_2	14	4
A_3	15	4
B_3	21	4
A_4	24	5

Lie Algebra	Dim	# Extr Elts
D_4	28	4
B_4	36	5
D_5	45	5
F_4	52	5
E_6	78	5
E_7	133	5
E_8	248	5

Five Generators - 3

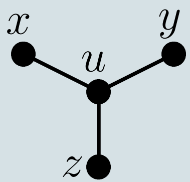
The biggest semi-simple Lie algebra generated by five extremal elements is E_8, of dimension 248!

# Extr Elts	Dimension	General Lie algebra
1	1	n/a (0 dimensional)
2	3	A_1 (3 dimensional)
3	8	A_2 (8 dimensional)
4	28	D_4 (28 dimensional)
5	537	E_8 (248 dimensional)

$$0, 3, 8, 28, 248$$

Examples - Introduction

Consider 4 and 5 generator cases in more detail:



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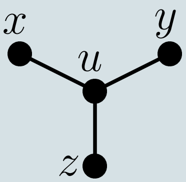

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- $\dim(L) = 12$ and $\dim(\mathrm{Rad}(L)) = 9$.

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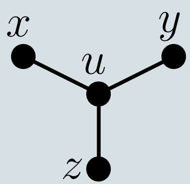

- 4 extremal elements generating L,
- $[x, y] = [x, z] = [y, z] = 0$,
- $[x, u], [y, u], [z, u] \neq 0$.
- $\dim(L) = 12$ and $\dim(\mathrm{Rad}(L)) = 9$,
- $\dim(L/\mathrm{Rad}(L)) = 12 - 9 = 3$,
- so $L/\mathrm{Rad}(L) \cong A_1$.

Examples - Four generators

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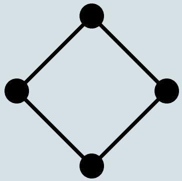

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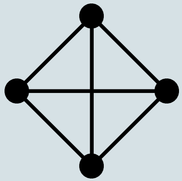
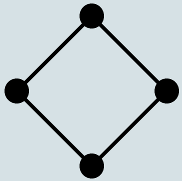


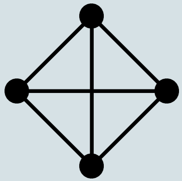
28

Examples - Four generators

10
C_2

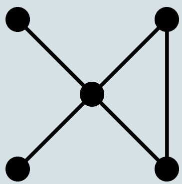


15
A_3

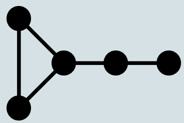


28
D_4

Examples - Five generators



36



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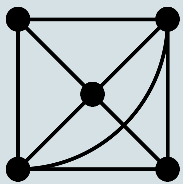
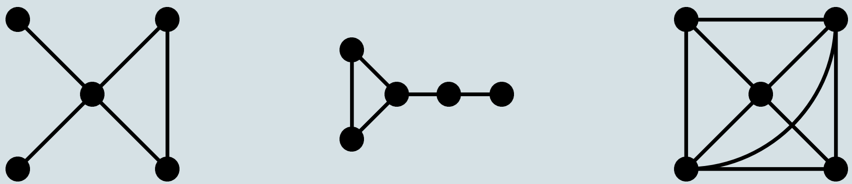


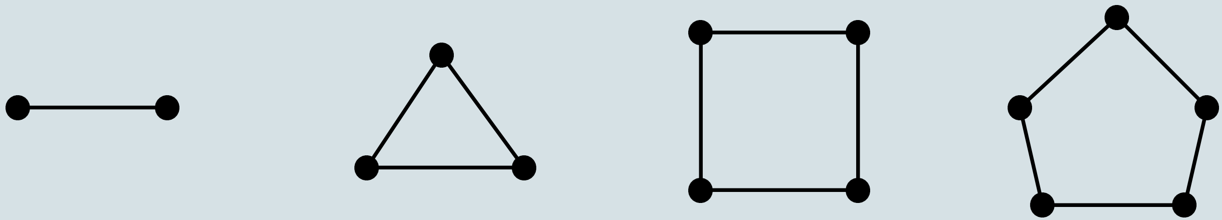
249

Examples - Five generators



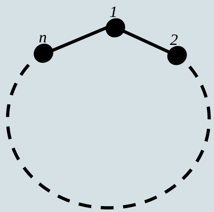
36 B_4 | 24 A_4 | 249 E_8 ?

Three Isomorphisms



3	8	15	24
A_1	A_2	A_3	A_4

Three Isomorphisms



isomorphic to A_{n-1}.

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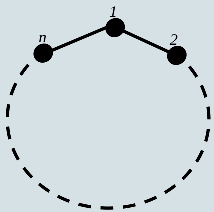

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isomorphic to C_n.

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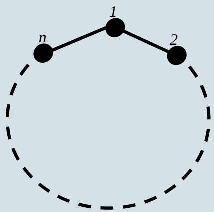

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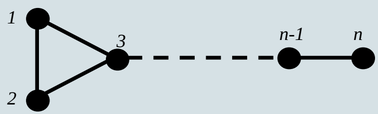
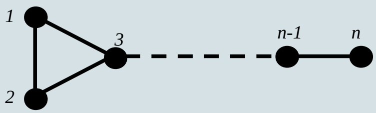
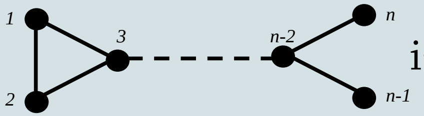


almost isomorphic to A_{n-1}.

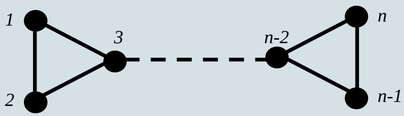
Three Conjectures



really isomorphic to A_{n-1}.



isomorphic to B_{n-1}.



isomorphic to D_n.

Conclusion

- Considered four and five generator case
- Proved semi-simplicity theorem
- Proved three isomorphisms
- Found three conjectures

Future research

- How can we use extremal elements?
- Prove three conjectures
- $1, 3, 8, 28, 537, \ldots$

Questions?

References

[ZK90] E. I. Zel′manov and A. I. Kostrikin. A theorem on sandwich algebras. *Trudy Mat. Inst. Steklov.*, 183:106–111, 225, 1990. Translated in Proc. Steklov Inst. Math. **1991**, no. 4, 121–126, Galois theory, rings, algebraic groups and their applications (Russian).