Lie Algebras
generated by
Extremal Elements

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Lie Algebras - 1

A Lie algebra L is a vector space with a map $[\cdot,\cdot] : V \times V \to V$ such that

1. $[\cdot,\cdot]$ is bilinear,
2. $[\cdot,\cdot]$ is skew-symmetric:

$$[x,x] = 0 \text{ for all } x \in L, \text{ and}$$

3. $[\cdot,\cdot]$ satisfies the Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \text{ for all } x,y,z \in L.$$

Note that in characteristic not 2, property 2. is equivalent to $[x,y] = -[y,x]$.

Lie Algebras - 2: An example

An example: $\mathfrak{gl}(V)$.

Let V be a vector space, and L the ring of linear transformations $V \to V$.
Define $[x, y] = xy - yx$ for $x, y \in L$.

Lie Algebras - 3: Another example

The special linear Lie algebra $\mathfrak{sl}_2$:

$$\text{Let } x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $[x, y] = xy - yx$. Then

$$z = [x, y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $[x, z] = -2x$ and $[y, z] = 2y$.

Lie Algebras - 4: Ideals

- Ideal: I is an ideal of L if

$$x \in L, y \in I \Rightarrow [x, y] \in I.$$

- Simplicity: L is called simple if it has no ideals except $\{0\}$ and L.
- Semi-simplicity

Lie Algebras - 5: Ideals

- Solvability: I is solvable if the sequence

$$L, [L, L], [[L, L], [L, L]], [[[L, L], [L, L]], [[L, L], [L, L]]], \ldots$$

  'goes to zero'.
- Radical: $\mathrm{Rad}(L)$ is the largest solvable ideal.
- Theorem: $L/\mathrm{Rad}(L)$ is semi-simple.

Lie Algebras - 6: Simple Lie algebras

The classical Lie algebras:

- A_n $(n \geq 1)$: $\mathfrak{sl}_{n+1}$, of dimension $(n+1)^2 - 1$,
- B_n $(n \geq 2)$: $\mathfrak{so}_{2n+1}$, of dimension $n(2n+1)$,
- C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}$, of dimension $n(2n+1)$,
- D_n $(n \geq 4)$: $\mathfrak{so}_{2n}$, of dimension $n(2n-1)$.

The special Lie algebras: G_2, F_4, E_6, E_7, E_8.

Extremal Elements - 1: Definition

- $x \in L$ is called *extremal* if $[x, [x, L]] \subseteq Kx$.
- There exists a linear functional f_x such that $[x, [x, y]] = f_x(y)x$.

Extremal Elements - 2: Example

The special linear Lie algebra $\mathfrak{sl}_2$:

$$\text{Let } x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $[x, y] = xy - yx$. Then

$$z = [x, y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $[x, [x, y]] = [x, z] = -2x$ and $[y, [y, x]] = -[y, z] = -2y$.

‘$\mathfrak{sl}_2$ is generated by two extremal elements’

Some results of CSUW01 - 1

Cohen, Steinbach, Ushirobira, Wales, 2001. Suppose L is generated by finitely many extremal elements. Then:

- L is finite dimensional, and
- L is spanned by extremal elements.

(Using [ZK90])

Some results of CSUW01 - 2

#	Dimension	General Lie algebra
2	3	A_1
3	8	A_2
4	28	?
5	537	?

Some results of CSUW01 - 3

Simple Lie algebra	Required extremal elements	Condition
A_n	$n+1$	$n \geq 1$
B_n	$n+1$	$n \geq 3$
C_n	$2n$	$n \geq 2$
D_n	n	$n \geq 4$
E_6, E_7, E_8, F_4	5	
G_2	4	

Some results of CSUW01 - 4

Suppose L is generated by finitely many extremal elements. Then:

- There exists a bilinear functional $f : L \times L \to K$ with $f(x, y) = f_x(y)$.
- f is associative: $f(x, [y, z]) = f([x, y], z)$.
- $\mathrm{Rad}(L) = \{x \in L \mid f_x = 0\}$, provided $\mathrm{char}(K) \neq 2, 3$.

(Last result owed to Gabor Ivanyos)

Two algorithms - 1

Given n, the number of extremal generators, and possibly pairs (i, j) such that $[i, j] = 0$:

1. Calculate (more or less monomial) basis for L:
   - Suppose $f = 0$,
   - Use graded Gröbner basis algorithm [GBNP] to obtain a basis,
   - Track process to obtain monomial basis,
   - Written in GAP.
2. Find evaluation of f on these basis elements.

Two algorithms - 2

Given n, the number of extremal generators, and possibly pairs (i, j) such that $[i, j] = 0$:

1. Calculate (more or less monomial) basis for L:
2. Find evaluation of f on these basis elements.
   - Input: Basis from the first step,
   - Try to apply rewrite rules and Jacobi identity,
   - Find 'primitive evaluations' along the way,
   - Written in C++.

Two algorithms - 3

Given n, the number of extremal generators, and possibly pairs (i, j) such that $[i, j] = 0$:

1. Calculate (more or less monomial) basis for L:
2. Find evaluation of f on these basis elements.

Use the result of the second step to calculate $\mathrm{Rad}(f)$, and the dimension of the semi-simple part of L.

A Theorem - 1

Suppose the semi-simple Lie algebra L_1 is generated (as a Lie algebra) by n extremal elements and no fewer, and the semi-simple Lie algebra L_2 is generated (as a Lie algebra) by m extremal elements and no fewer. Then the semi-simple Lie algebra $L_1 + L_2$ is generated by $n + m$ extremal elements and no fewer.

A Theorem - 2: Sketch of the proof

$L = L_1 + L_2$. Suppose to the contrary that L is generated by less than $n+m$ extremal elements.

- Extremal basis elements $x_1, \ldots, x_N$ and $y_1, \ldots, y_M$.
- L has a basis of extremal elements. wlog $z = x_1 + y_1$ is among them. Then:

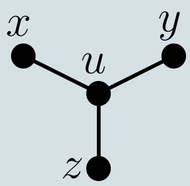
$$f(x_1 + y_1, x_2)z = f(z, x_2)z = [z, [z, x_2]] = \ldots = f(x_1, x_2)x_1.$$

  hence $f(z, x_2) = 0$. Some reasoning shows $f_z = 0$.
- So $\dim(\mathrm{Rad}(f)) \geq 1$. Contradiction.

A Theorem - 3: Corollary

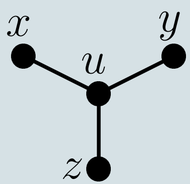
No 537 dimensional Lie algebra generated by five extremal elements exists.

Examples - Introduction



- 4 extremal generators,
- $[x, y] = [x, z] = [y, z] = 0$,
- $< x, u >, < y, u >, < z, u > \cong \mathfrak{sl}_2$.

Examples - Introduction



- 4 extremal generators,
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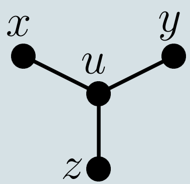
- 12-dimensional

Examples - Introduction



- 4 extremal generators,
- $[x, y] = [x, z] = [y, z] = 0$,
- $< x, u >, < y, u >, < z, u > \cong \mathfrak{sl}_2$.
- 12-dimensional
- $\dim(\mathrm{Rad}(f)) = 9$

Examples - Four generators

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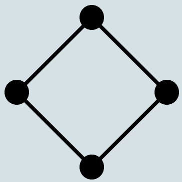

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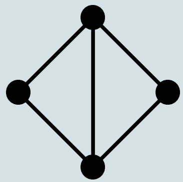

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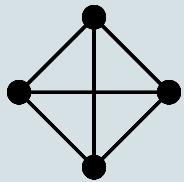

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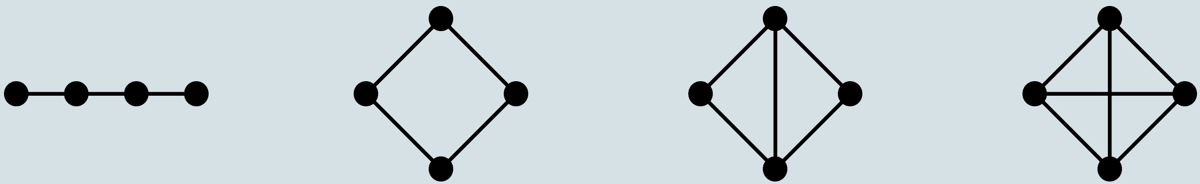
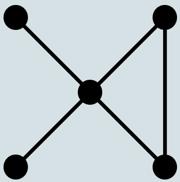
Examples - Four generators



10 C_2 | 15 A_3 | 21 B_3 | 28 D_4

Examples - Five generators



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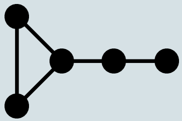

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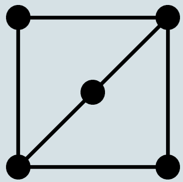

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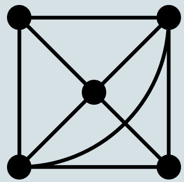

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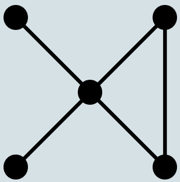
Examples - Five generators



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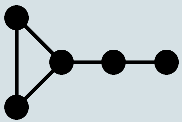
B_4



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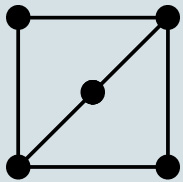
A_4



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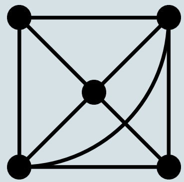
F_4 ?



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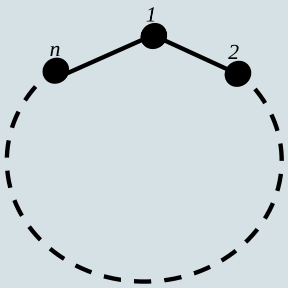
E_8 ?

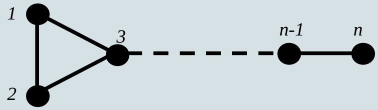
Two Theorems



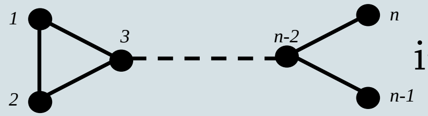
isomorphic to A_{n-1}.

isomorphic to C_n.

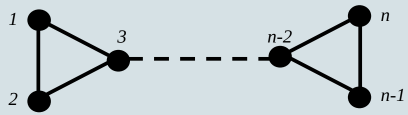
Three Propositions



isomorphic to A_{n-1}.



isomorphic to B_{n-1}.



isomorphic to D_n.

Conclusion

Done:

- Verified some results from [CSUW01]
- Considered degenerate cases
- Proved semi-simplicity theorem
- Proved two isomorphisms

To do in the remaining six weeks:

- Prove three propositions
- Improve second algorithm
- Finish report

Questions?

References

[CSUW01] A. M. Cohen, A. Steinbach, R. Ushirobira, and D. Wales. Lie algebras generated by extremal elements. *J. Algebra*, 236(1):122–154, 2001.

[ZK90] E. I. Zel′manov and A. I. Kostrikin. A theorem on sandwich algebras. *Trudy Mat. Inst. Steklov.*, 183:106–111, 225, 1990. Translated in Proc. Steklov Inst. Math. **1991**, no. 4, 121–126, Galois theory, rings, algebraic groups and their applications (Russian).